Lecture 3 — First-Order Theories

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  - e.g., $I = \{\mathbb{N}, \alpha_I\}$ s.t. $\alpha_I(+) = \{\dots, (1,1) \mapsto 3, \dots\}$
- We wish to restrict possible interpretations of $\varphi \rightsquigarrow$ **theories**.

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  - often, we need an *infinite* number of axioms $\rightsquigarrow$ axiom schemata
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- e.g., the quantifier-free fragment, alternation-free fragment, existential fragment, fragments restricting the number of quantifier alternations, . . .
- we often show equiv. of (fragments of) theories with other formal models

$\mathcal{T}$-validity and $\mathcal{T}$-satisfiability

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- $\mathcal{T}$-interpretation: an interpretation I that satisfies all axioms of $\mathcal{T}$:

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- A $\Sigma_{\mathcal{T}}$-formula φ is $\mathcal{T}$-satisfiable if it holds for some $\mathcal{T}$-interpretation
  - SMT: satisfiability modulo theories
  - SMT-solvers: programs "deciding" $\mathcal{T}$-satisfiability of formulae
    - i.e., deciding for decidable (fragments of) theories
    - trying to decide for undecidable

Interpretations

Indistinguishability:

- A ground formula φ distinguishes two interpretations I_1 and I_2 if

$$I_1 \models \varphi \qquad \text{if and only if} \qquad I_2 \not\models \varphi$$

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- I_1 and I_2 are indistinguishable if there is no FOL formula that distinguishes them:
  - isomorphic models
  - others (we'll see later)

Negation Completeness and Consistency

Negation Completeness:

- A theory $\mathcal{T}$ is negation complete if for every *closed* $\Sigma_{\mathcal{T}}$-formula φ,

  it holds that $\quad \mathcal{T} \vdash \varphi \quad$ or $\quad \mathcal{T} \vdash \neg\varphi$

  ($\mathcal{T} \vdash \varphi$ means "φ is provable in $\mathcal{T}$").
  - Can be seen as whether the axiomatization sufficiently restricts interpretations
    - i.e., all models of $\mathcal{T}$ are indistuiguishable by any $\mathcal{T}$-formula
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  - (A proof system S for FOL is *complete* if for every FOL formula φ such that $\models \varphi$, it holds that $\vdash_S \varphi$.)

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- Alternative definition: A theory is inconsistent if for every $\Sigma_{\mathcal{T}}$-formula φ it holds that $\mathcal{T} \vdash \varphi$, otherwise it is consistent.

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  - validity: when testing validity, a quantifier-free formula is prefixed by universal quantification of free variables

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- here, we consider it a part of FOL($\emptyset$) (i.e., it is used implicitly in the other theories)
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  1. $\forall x (x = x)$ (reflexivity)
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- Note that [4] and [5] are *axiom schemata*.

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Example

The formula

$$f(f(f(a))) = a \quad \wedge \quad f(f(f(f(f(a))))) = a \quad \wedge \quad f(a) \neq a$$

is unsatisfiable.

Intermission — Numbers

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  2. $\forall x \forall y (S(x) = S(y) \quad \rightarrow \quad x = y)$ (injectivity of S)
  3. for every $\Sigma_{\mathcal{T}_{\mathsf{PA}}}$-formula φ with precisely one free variable,

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Peano Arithmetic $\mathcal{T}_{\mathsf{PA}}$

Peano Arithmetic $\mathcal{T}_{\mathsf{PA}}$ (first-order arithmetic):

- Signature: $\langle \mathcal{F} = \{0_{/0}, S_{/1}, +_{/2}, \cdot_{/2}\}, \mathcal{P} = \emptyset \rangle$
- Axioms:
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$$\exists x \exists y \exists z(x \neq 0 \land y \neq 0 \land z \neq 0 \land (x \cdot x) + (y \cdot y) = (z \cdot z))$$

Peano Arithmetic $\mathcal{T}_{\mathsf{PA}}$

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- recursive theory: there is an algorithm that will, given a formula φ, decide whether φ is an axiom of the theory
  - all commonly considered theories are recursive

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$$\text{neither} \qquad \mathcal{T}_{PA} \vdash \varphi \qquad \text{nor} \qquad \mathcal{T}_{PA} \vdash \neg\varphi$$

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- therefore, every sufficiently strong formal system (in particular, a system with arithmetic) is either inconsistent or negation incomplete

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Proof. (high-level idea).

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Let $\Sigma = \{a, b, c\}$ and let $\# : \Sigma \to \mathbb{N}$ be injective, e.g., $\#(a) = 2, \#(b) = 3, \#(c) = 4$. Then the number

$$2^{\#(a)} \cdot 3^{\#(b)} \cdot 5^{\#(c)} \cdot 7^{\#(b)} \cdot 11^{\#(a)} = 2{,}801{,}452{,}500$$

uniquely encodes the string "$abcba$".

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Peano Arithmetic $\mathcal{T}_{\mathsf{PA}}$

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"φ is true iff φ is unprovable."
- Generalization of the "Liar's paradox." (diagonalization)

□

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Theorem (Gödel's Completeness Theorem)

FOL with the semantic argument proof system is complete.

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- Therefore, by G's Completeness T., there are models of PA where GS is true and models where it is false.
- $\leadsto$ there exist **nonstandard models of Peano Arithmetic**

Presburger Arithmetic $\mathcal{T}_{\mathbb{N}}$

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The following formula over $\mathbb{Z}$

$$\forall x \forall z \exists y (2x - y = 3z + 5)$$

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$$\forall x_p \forall x_n \forall z_p \forall z_n \exists y_p \exists y_n \big(\, 2(x_p - x_n) - (y_p - y_n) = 3(z_p - z_n) + 5 \, \big).$$

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Expressed in $\mathcal{T}_{\mathbb{N}}$ by moving negative terms to the other side:

$$\forall x_p \forall x_n \forall z_p \forall z_n \exists y_p \exists y_n (2x_p + y_n + 3z_n = 3z_p + 5 + 2x_n + y_p).$$

Theory of Integers $\mathcal{T}_{\mathbb{Z}}$

Theory of Integers $\mathcal{T}_{\mathbb{Z}}$:

- Signature:
  $\langle \mathcal{F} = \{\ldots, -2_{/0}, -1_{/0}, 0_{/0}, 1_{/0}, 2_{/0}, \ldots, (-2\cdot)_{/1}, (2\cdot)_{/1}, (3\cdot)_{/1}, \ldots, +_{/2}, -_{/2}\}, \mathcal{P} = \{<_{/2}\}\rangle$

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- Every $\Sigma_{\mathcal{T}_{\mathbb{Z}}}$-formula can be reduced to $\Sigma_{\mathcal{T}_{\mathbb{N}}}$.

Theory of Reals $\mathcal{T}_{\mathbb{R}}$

Theory of Reals $\mathcal{T}_{\mathbb{R}}$ (elementary algebra):

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- Axioms of an **abelian group**:
  1. $\forall x \forall y \forall z ((x + y) + z = x + (y + z))$ (+ associativity)
  2. $\forall x (x + 0 = x)$ (+ identity)
  3. $\forall x (x + (-x) = 0)$ (+ inverse)
  4. $\forall x \forall y (x + y = y + x)$ (+ commutativity)

Theory of Reals $\mathcal{T}_{\mathbb{R}}$

- Additional axioms of a **ring**:
  1. $\forall x \forall y \forall z \big((x \cdot y) \cdot z = x \cdot (y \cdot z)\big)$ ($\cdot$ associativity)
  2. $\forall x (x \cdot 1 = x)$ ($\cdot$ right identity)
  3. $\forall x (1 \cdot x = x)$ ($\cdot$ left identity)
  4. $\forall x \forall y \forall z \big(x \cdot (y + z) = (x \cdot y) + (x \cdot z)\big)$ ($\cdot$ left distributivity over $+$)
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- Additional axioms of a **field**:
  1. $\forall x \forall y (x \cdot y = y \cdot x)$ ($\cdot$ commutativity)
  2. $0 \neq 1$ (separate identities)
  3. $\forall x \big(x \neq 0 \quad \rightarrow \quad \exists y (x \cdot y = 1)\big)$ ($\cdot$ inverse)

Theory of Reals $\mathcal{T}_{\mathbb{R}}$

- Axioms of a **total order**:
  1. $\forall x \forall y ((x \leq y \wedge y \leq x) \quad \rightarrow \quad x = y)$ (antisymmetry)
  2. $\forall x \forall y \forall z \big((x \leq y \wedge y \leq z) \quad \rightarrow \quad x \leq z\big)$ (transitivity)
  3. $\forall x \forall y (x \leq y \vee y \leq x)$ (totality)

Theory of Reals $\mathcal{T}_{\mathbb{R}}$

- Axioms of a **total order**:
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  2. $\forall x \forall y \forall z \big((x \leq y \land y \leq z) \quad \rightarrow \quad x \leq z\big)$ (transitivity)
  3. $\forall x \forall y (x \leq y \lor y \leq x)$ (totality)

- Additional axioms of a **real closed field**:
  1. $\forall x \forall y \forall z (x \leq y \quad \rightarrow \quad x + z \leq y + z)$ ($+$ ordered)
  2. $\forall x \forall y \big((0 \leq x \land 0 \leq y) \quad \rightarrow \quad 0 \leq x \cdot y\big)$ ($\cdot$ ordered)
  3. $\forall x \exists y (x = y^2 \lor x = -y^2)$ (square root)
  4. for every odd integer n,

$$\forall \overline{x} \exists y \big(y^n + (x_1 \cdot y^{n-1}) + \cdots + (x_{n-1} \cdot y) + x_n = 0\big)$$ (at least one root)

Theory of Reals $\mathcal{T}_{\mathbb{R}}$

- **decidable** [Tarski 1956]
  - via quantifier elimination

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Example

Can you find a quantifier-free formula $\mathcal{T}_{\mathbb{R}}$-equivalent to the formula

$$\exists x(ax^2 + bx + c = 0)?$$

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Theory of Reals $\mathcal{T}_{\mathbb{R}}$

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Example

Can you find a quantifier-free formula $\mathcal{T}_{\mathbb{R}}$-equivalent to the formula

$$\exists x(ax^2 + bx + c = 0)?$$

Solution: the formula

$$b^2 - 4ac \geq 0.$$

Theory of Rationals $\mathcal{T}_{\mathbb{Q}}$

Theory of Rationals $\mathcal{T}_{\mathbb{Q}}$:

- Signature: $\langle \mathcal{F} = \{0_{/0}, 1_{/0}, +_{/2}, -_{/1}\}, \mathcal{P} = \{\leq_{/2}\}\rangle$

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  3. $\forall x \forall y (x \leq y \lor y \leq x)$ (totality)
  4. $\forall x \forall y \forall z \big((x + y) + z = x + (y + z)\big)$ ($+$ associativity)
  5. $\forall x (x + 0 = x)$ ($+$ identity)
  6. $\forall x (x + (-x) = 0)$ ($+$ inverse)

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  7. $\forall x \forall y (x + y = y + x)$ ($+$ commutativity)
  8. $\forall x \forall y \forall z (x \leq y \quad \rightarrow \quad x + z \leq y + z)$ ($+$ ordered)

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  8. $\forall x \forall y \forall z (x \leq y \quad \rightarrow \quad x + z \leq y + z)$ ($+$ ordered)
  9. for each positive integer n,

$$\forall x (nx = 0 \quad \rightarrow \quad x = 0)$$ (torsion-free)

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  8. $\forall x \forall y \forall z (x \leq y \quad \rightarrow \quad x + z \leq y + z)$ ($+$ ordered)
  9. for each positive integer n,
     $$\forall x (nx = 0 \quad \rightarrow \quad x = 0)$$ (torsion-free)
  10. for each positive integer n,
     $$\forall x \exists y (x = ny)$$ (divisible)

  where nx denotes $\overbrace{x + \cdots + x}^{n}$

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Example

The formula

$$\frac{1}{2}x + \frac{2}{3}y \geq 4$$

can be expressed as the $\Sigma_{\mathcal{T}_{\mathbb{Z}}}$-formula

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The formula

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can be expressed as the $\Sigma_{\mathcal{T}_{\mathbb{Z}}}$-formula

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Example

The formula

$$\exists x (x \cdot x = 2)$$

is a valid formula of $\mathcal{T}_{\mathbb{R}}$ but is expressible in neither $\mathcal{T}_{\mathbb{Q}}$ nor $\mathcal{T}_{\mathbb{Z}}$.

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Example

The formula
$$\exists x(x \cdot x = 2)$$
is a valid formula of $\mathcal{T}_{\mathbb{R}}$ but is expressible in neither $\mathcal{T}_{\mathbb{Q}}$ nor $\mathcal{T}_{\mathbb{Z}}$.

Example

The formula
$$\forall x \forall y\big(x < y \quad \rightarrow \quad \exists z(x < z \land z < y)\big)$$
is a valid formula of $\mathcal{T}_{\mathbb{R}}$ and $\mathcal{T}_{\mathbb{Q}}$, but an invalid formula of $\mathcal{T}_{\mathbb{N}}$ and $\mathcal{T}_{\mathbb{Z}}$.

Theory of Lists $\mathcal{T}_{\mathsf{List}}$

Theory of Lists $\mathcal{T}_{\mathsf{List}}$:

- Signature: $\langle \mathcal{F} = \{\mathsf{cons}_{/2}, \mathsf{car}_{/1}, \mathsf{cdr}_{/1}\}, \mathcal{P} = \{\mathsf{atom}_{/1}\}\rangle$
  - $\mathsf{cons}_{/2}$ is a function symbol called the constructor
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  2. instantiations of the (function congruence) axiom scheme of $\mathcal{T}_E$:

$$\forall x_1 \forall x_2 \forall y_1 \forall y_2 \big((x_1 = x_2 \wedge y_1 = y_2) \quad \rightarrow \quad \text{cons}(x_1, y_1) = \text{cons}(x_2, y_2)\big)$$
$$\forall x \forall y (x = y \quad \rightarrow \quad \text{car}(x) = \text{car}(y))$$
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$$\forall x \forall y \big(x = y \quad \rightarrow \quad (\text{atom}(x) \leftrightarrow \text{atom}(y))\big)$$

  4. $\forall x \forall y (\text{car}(\text{cons}(x, y)) = x)$ (left projection)
  5. $\forall x \forall y (\text{cdr}(\text{cons}(x, y)) = y)$ (right projection)

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  6. $\forall x (\neg \text{atom}(x) \quad \rightarrow \quad \text{cons}(\text{car}(x), \text{cdr}(x)) = x)$ (construction)

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Theory of Lists $\mathcal{T}_{\text{List}}$

- **undecidable**

Theory of Lists $\mathcal{T}_{\text{List}}$

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- Theory of Acyclic Lists $\mathcal{T}^{+}_{\text{List}}$:
  - created by adding the following axiom schema:

$$
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\forall x(\mathsf{car}(x) \neq x) \\
\forall x(\mathsf{cdr}(x) \neq x) \\
\forall x(\mathsf{car}(\mathsf{car}(x)) \neq x) \\
\forall x(\mathsf{car}(\mathsf{cdr}(x)) \neq x) \\
\ldots
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$$

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- the quantifier-free fragment is **decidable**

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$$\ldots$$

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- a more general Theory of Recursive Data Structures available

Theory of Arrays $\mathcal{T}_{\mathsf{A}}$

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  1. (reflexivity), (symmetry), and (transitivity) of $\mathcal{T}_E$
  2. $\forall a \forall i \forall j (i = j \quad \rightarrow \quad a[i]^r = a[j]^r)$ (array congruence)

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  3. $\forall a \forall v \forall i \forall j \big(i = j \quad \rightarrow \quad (a[i,v]^w)[j]^r = v\big)$ (read over write 1)

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  3. $\forall a \forall v \forall i \forall j \big(i = j \quad \rightarrow \quad (a[i,v]^w)[j]^r = v\big)$ (read over write 1)
  4. $\forall a \forall v \forall i \forall j \big(i \neq j \quad \rightarrow \quad (a[i,v]^w)[j]^r = a[j]^r\big)$ (read over write 2)

Theory of Arrays $\mathcal{T}_\mathsf{A}$

- **undecidable**
  - arbitrary functions can be encoded using multi-dimensional arrays

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  - arbitrary functions can be encoded using multi-dimensional arrays
- extended with the (extensionality) axiom, the quantifier-free fragment is decidable

$$\forall a \forall b \big(\forall i (a[i]^r = b[i]^r) \quad \leftrightarrow \quad a = b\big) \qquad \text{(extensionality)}$$

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Example

The formula

$$a[i]^r = e \quad \rightarrow \quad \forall j \big((a[i, e]^w)[j]^r = a[j]^r \big)$$

is $\mathcal{T}_\mathsf{A}$-valid.

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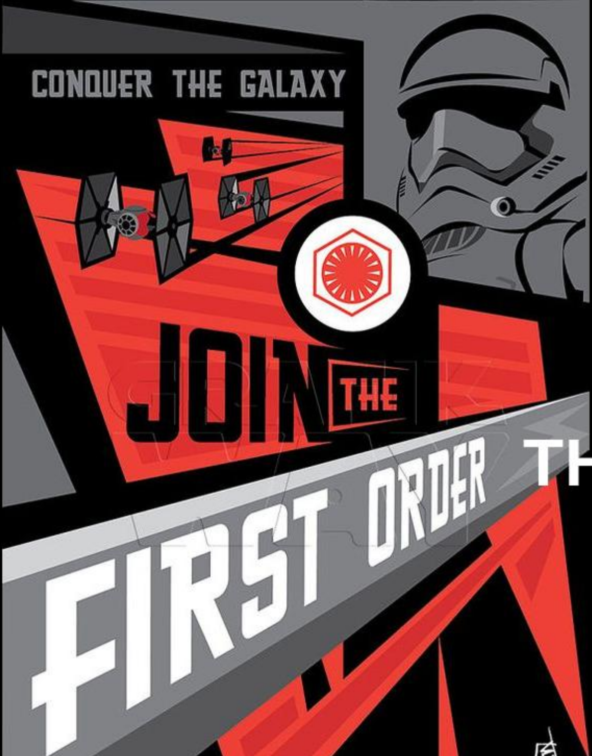

THEORIES