Lazy Automata Techniques for WS1S

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TACAS'17

WS1S

- weak monadic second-order logic of one successor
  - second-order ⇒ quantification over relations;
  - monadic ⇒ relations are unary (i.e. sets);
  - weak ⇒ sets are finite;
  - of one successor ⇒ reasoning about linear structures.

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- decidable — but **NONELEMENTARY**
  - constructive proof via translation to finite automata

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- decision procedure: the well-known MONA tool
  - sometimes efficient in practice
  - other times the complexity strikes back (unavoidable in general)
  - we try to push the usability border further!!

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- Interpretation: over finite subsets of $\mathbb{N}$
  - models of formulae = assignments of sets to variables
- sets can be encoded as binary strings:
  - $\{1, 4, 5\} \rightarrow$

Index:	012345	,	012345|6	or	012345|67	...
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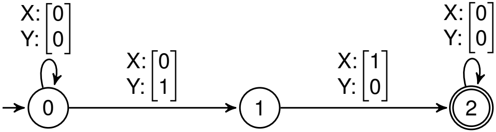
  - **Models** are represented as a stack of (0-padded) binary strings
  - Example:

$$\{X \mapsto \emptyset, Y \mapsto \{2,4\}\} \models \varphi \quad \text{iff} \quad \begin{matrix} X: \\ Y: \end{matrix}\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} \in L(\varphi)$$

Deciding WS1S using automata

- example of base automaton for $X = \sigma(Y)$ (successor)



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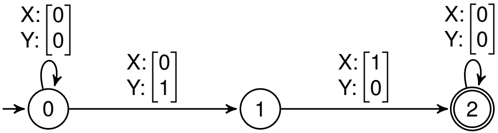

- Example:

$$\neg\underbrace{(X \subseteq Y)}_{\mathcal{A}_3} \wedge \Big(\underbrace{\mathrm{Sing}(Z)}_{\mathcal{A}_2} \vee \exists W. \underbrace{W = \sigma(Z)}_{\mathcal{A}_1}\Big)$$

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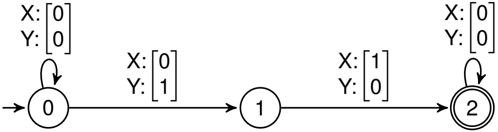

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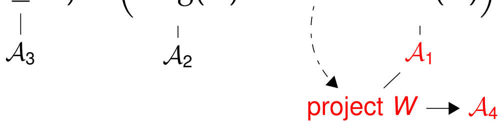

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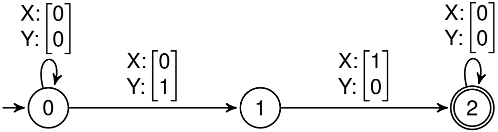

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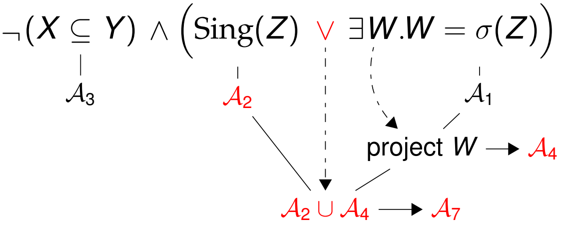

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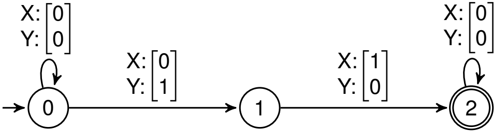

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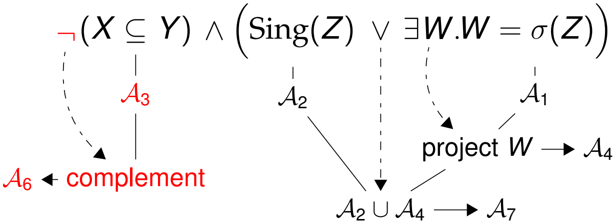

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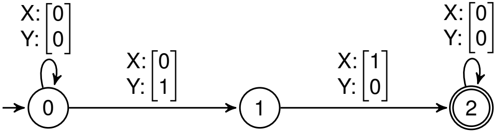

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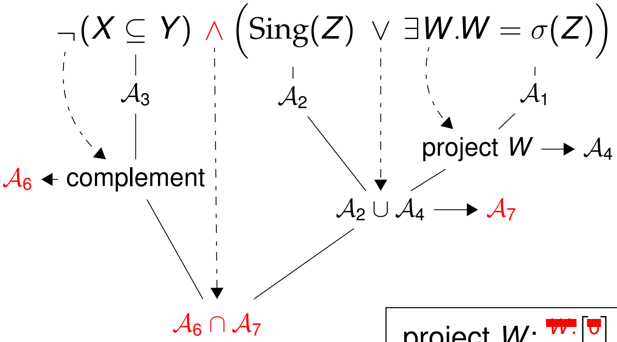

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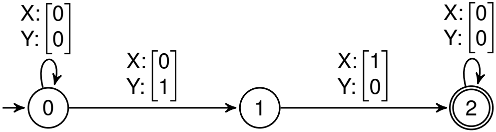

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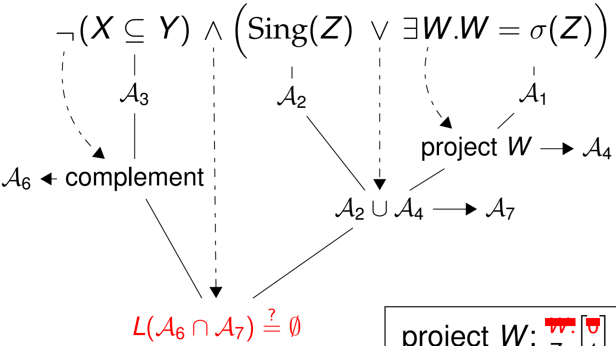

project W: ~~W:~~ Z: $\begin{bmatrix} \not 0 \\ 1 \end{bmatrix} \mapsto Z$: [1]

How to handle quantification

- issue with projection (existential quantification)
  - after removing of the tracks not all models would be accepted (problem with 0-padding)
    - needed for **soundness**!
    - it is necesssary to accept all or none encodings of the models
  - so after projection we need to adjust the final states by saturation
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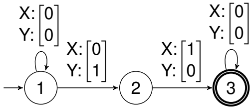

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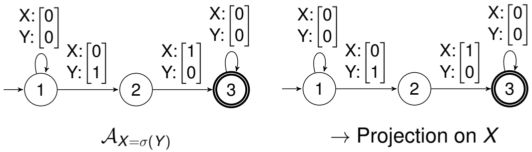

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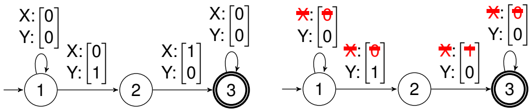

$\mathcal{A}_{X=\sigma(Y)}$ $\rightarrow$ Projection on X

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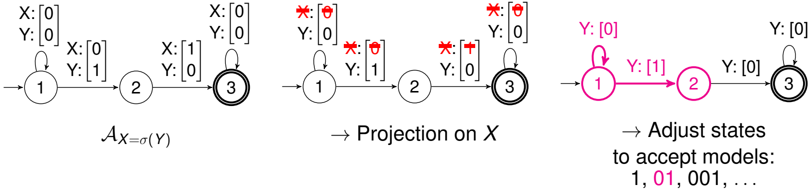

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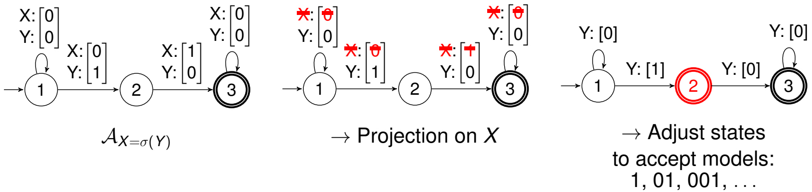

Ground Formulae

We focus on validity of ground formulae (all variables are quantified)

- satisfiability/validity of other formulae: prefixing with $\exists / \forall$

Key observation for ground formulae

$$\models \varphi \quad \text{iff} \quad \varepsilon \in L(\varphi)$$

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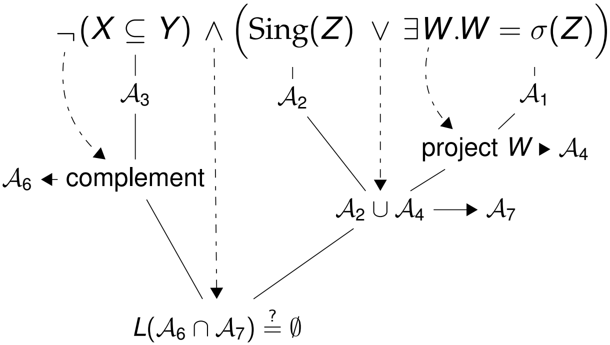
Why?

- Formula φ is valid if it accepts everything ($L(\varphi) = \Sigma^*$)
- Formula φ is unsatisfiable if it accepts nothing ($L(\varphi) = \emptyset$)
  - so it is sufficient to just test membership of ε

Problems with constructing automata



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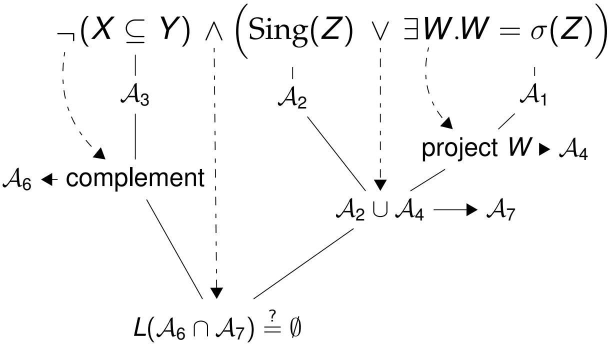

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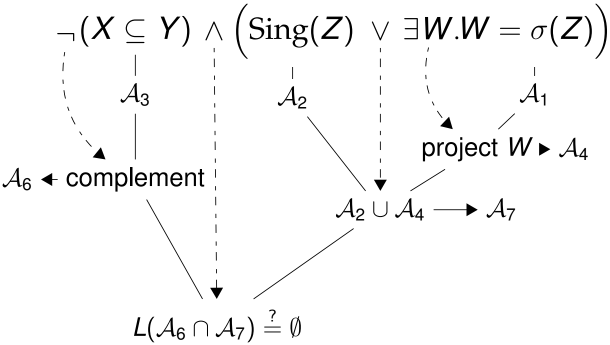

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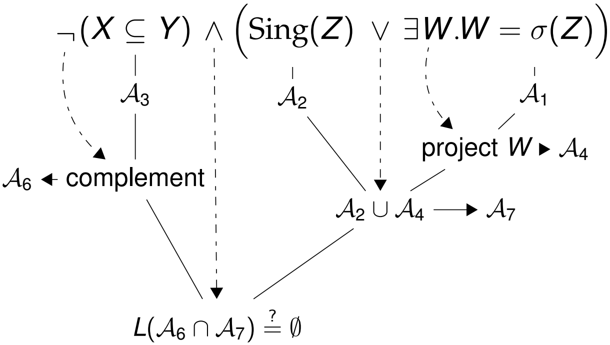

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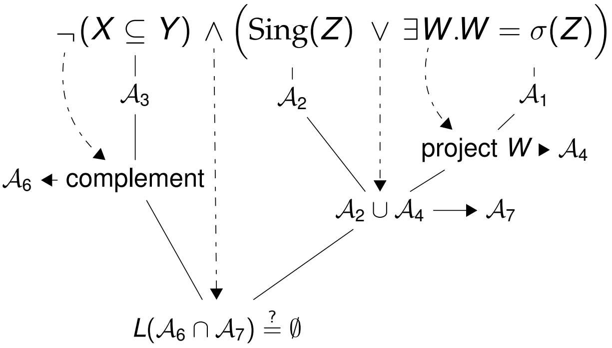
3. For $\mathcal{A}_1 \cap \mathcal{A}_2$, what if $L(\mathcal{A}_1) = \emptyset$?
   - No need to construct $\mathcal{A}_2$ and $\mathcal{A}_1 \cap \mathcal{A}_2$!

Towards Language Terms



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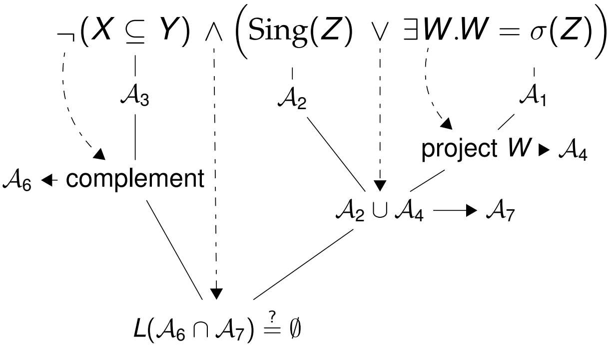

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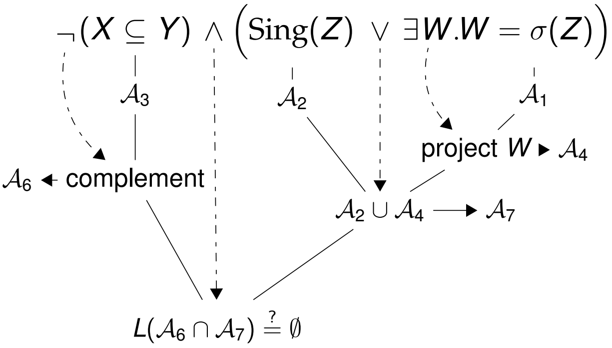

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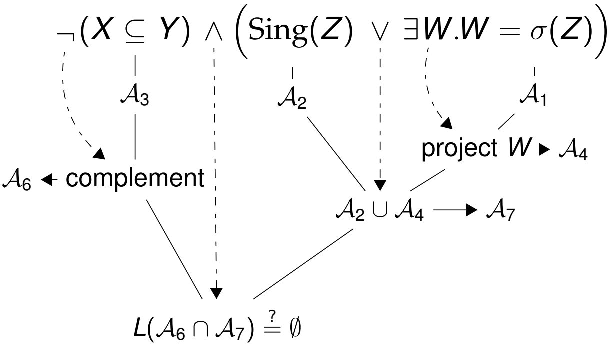

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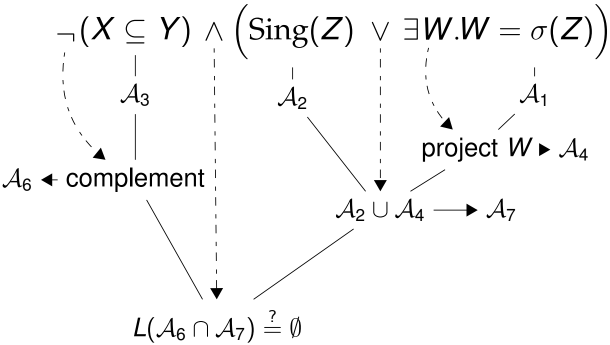

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  - Use subsumption to prune state space

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   - Intuition: Automaton either accepts Σ^* or nothing, so ε test suffices
   - $\models \varphi \iff \varepsilon \in t_\varphi$

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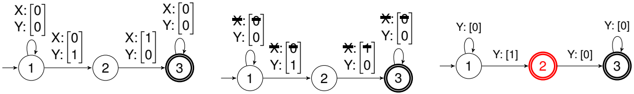
3 Lazy evaluation of ε-membership on term t

- $\varepsilon \in \mathcal{A} \Leftrightarrow I_{\mathcal{A}} \cap F_{\mathcal{A}} \neq \emptyset$
- $\varepsilon \in t_{\varphi} \cap t_{\psi} \Leftrightarrow \varepsilon \in t_{\varphi} \land \varepsilon \in t_{\psi}$
  - if $\varepsilon \notin t_{\varphi}$ no need to check if $\varepsilon \in t_{\psi}$
- $\varepsilon \in t_{\varphi} \cup t_{\psi} \Leftrightarrow \varepsilon \in t_{\varphi} \lor \varepsilon \in t_{\psi}$
  - if $\varepsilon \in t_{\varphi}$ no need to check if $\varepsilon \in t_{\psi}$
- $\varepsilon \in \overline{t_{\varphi}} \Leftrightarrow \varepsilon \notin t_{\varphi}$
- $\varepsilon \in \pi_X(t_{\varphi}) \Leftrightarrow \varepsilon \in t_{\varphi}$

Overview of our method

3 Lazy evaluation of ε-membership on term t

- $\varepsilon \in t - \overline{0}^* \Leftrightarrow \varepsilon \in t \vee \varepsilon \in t - \overline{0} \vee \varepsilon \in t - \overline{0}\,\overline{0} \vee \ldots$
  - evaluation of the quotients leads to fixpoint computations
  - lazy evaluation $\rightsquigarrow$ iteratively test $\varepsilon \in t, \varepsilon \in t - \overline{0}, \ldots$
  - ...until fixpoint reached or satisfying member found



$\mathcal{A}_{X=\sigma(Y)}$

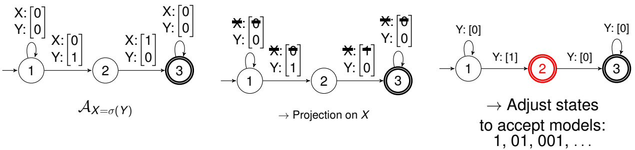
→ Projection on X

→ Adjust states to accept models: 1, 01, 001, ...

Overview of our method

3 Lazy evaluation of ε-membership on term t

- $\varepsilon \in t - \overline{0}^* \Leftrightarrow \varepsilon \in t \vee \varepsilon \in t - \overline{0} \vee \varepsilon \in t - \overline{0}\,\overline{0} \vee \ldots$
  - evaluation of the quotients leads to fixpoint computations
  - lazy evaluation $\rightsquigarrow$ iteratively test $\varepsilon \in t, \varepsilon \in t - \overline{0}, \ldots$
  - ...until fixpoint reached or satisfying member found
- $\varepsilon \in t - \overline{0}$
  - $-\overline{0}$ on inner nodes: push through to leaves
  - $-\overline{0}$ on leaves: compute 0-predecessors of final states



Overview of our method

3 Lazy evaluation of ε-membership on term t

- $\varepsilon \in t - \overline{0}^* \Leftrightarrow \varepsilon \in t \vee \varepsilon \in t - \overline{0} \vee \varepsilon \in t - \overline{0}\,\overline{0} \vee \ldots$
  - evaluation of the quotients leads to fixpoint computations
  - lazy evaluation $\rightsquigarrow$ iteratively test $\varepsilon \in t, \varepsilon \in t - \overline{0}, \ldots$
  - …until fixpoint reached or satisfying member found
- $\varepsilon \in t - \overline{0}$
  - $-\overline{0}$ on inner nodes: push through to leaves
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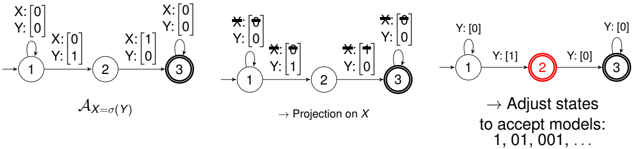
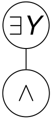


4 Further optimizations

- e.g. subsumption, continuations, formula preprocessing, etc.

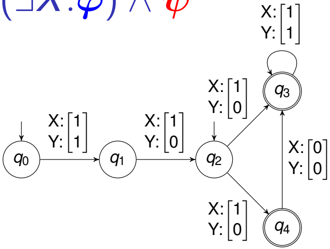
Validity checking of $\exists Y.(\exists X.\varphi) \land \psi$

Validity checking of $\exists Y.(\exists X.\varphi) \land \psi$



- We represent the formula symbolically as a language terms $t_{\exists Y.(\exists X.\varphi)\land\psi}$ and test the emptiness.
- $\varepsilon \in t_{\exists Y.(\exists X.\varphi)\land\psi} \iff \varepsilon \in t_{\exists X.\varphi} \cap t_{\psi} - \bar{0}^*$
  $\iff \varepsilon \in t_{\exists X.\varphi} \cap t_{\psi} \lor \varepsilon \in t_{\exists X.\varphi} \cap t_{\psi} - \bar{0} \lor \varepsilon \in t_{\exists X.\varphi} \cap t_{\psi} - \bar{0}^2 \ldots$
- We will demonstrate our method just on testing if $\varepsilon \in t_{\exists X.\varphi} \cap t_{\psi}$
  - (some details will be omitted)

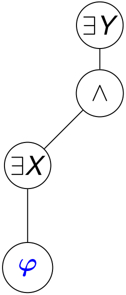
Validity checking of $\exists Y.(\exists X.\varphi) \land \psi$

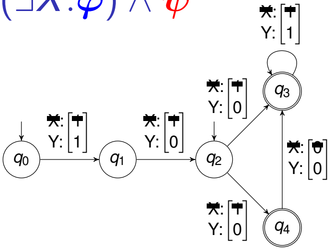


(a) Automaton for φ

- Term $t_{\exists X.\varphi}$ corresponds to the left subformula $\exists X.\varphi$

Validity checking of $\exists Y.(\exists X.\varphi) \wedge \psi$

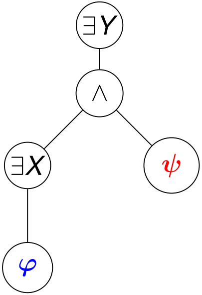


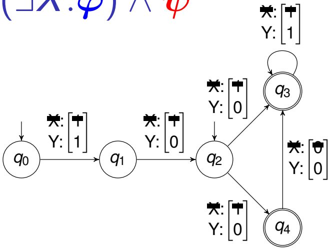


(a) Automaton for $\exists X.\varphi$

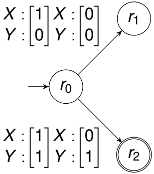
- Term $t_{\exists X.\varphi}$ corresponds to the left subformula $\exists X.\varphi$

Validity checking of $\exists Y.(\exists X.\varphi) \wedge \psi$





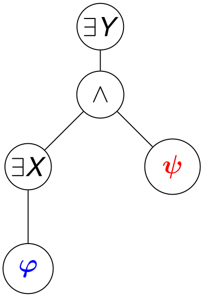
(a) Automaton for $\exists X.\varphi$

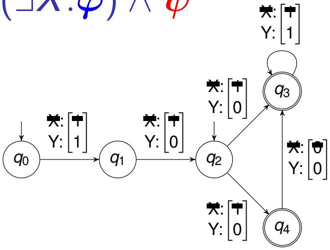


(b) Automaton for ψ

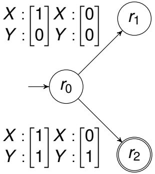
- Term $t_{\exists X.\varphi}$ corresponds to the left subformula $\exists X.\varphi$
- Term t_{ψ} corresponds to the right subformula ψ

Validity checking of $\exists Y.(\exists X.\varphi) \wedge \psi$





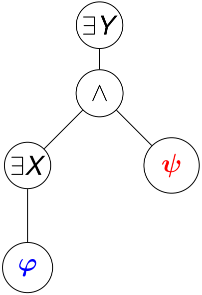
(a) Automaton for $\exists X.\varphi$

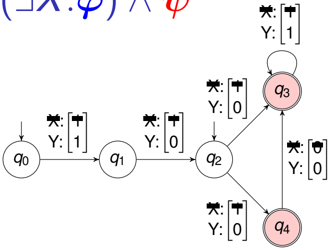


(b) Automaton for ψ

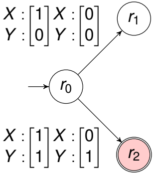
- We commence the emptiness check from final states of leaf automata.
- (After projection new final states are backward reachable from current final states)

Validity checking of $\exists Y.(\exists X.\varphi) \wedge \psi$





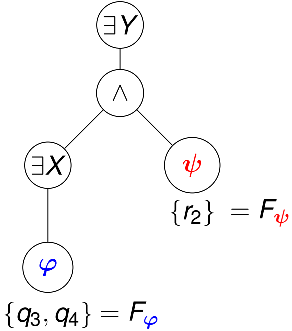
(a) Automaton for $\exists X.\varphi$

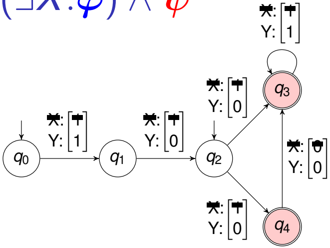


(b) Automaton for ψ

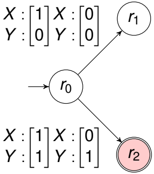
- We commence the emptiness check from final states of leaf automata.
- (After projection new final states are backward reachable from current final states)

Validity checking of $\exists Y.(\exists X.\varphi) \wedge \psi$





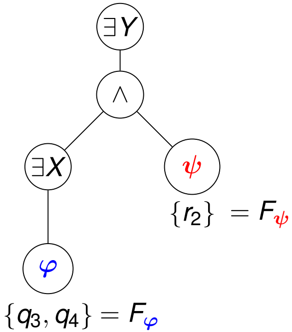
(a) Automaton for $\exists X.\varphi$

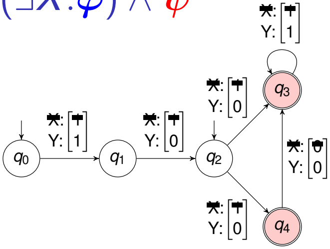


(b) Automaton for ψ

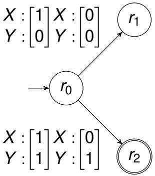
- We commence the emptiness check from final states of leaf automata.
- (After projection new final states are backward reachable from current final states)

Validity checking of $\exists Y.(\exists X.\varphi) \wedge \psi$





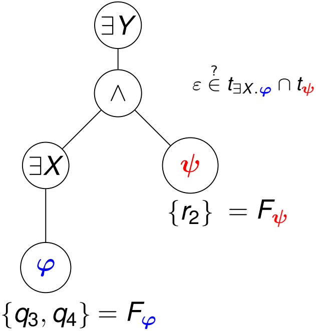
(a) Automaton for $\exists X.\varphi$

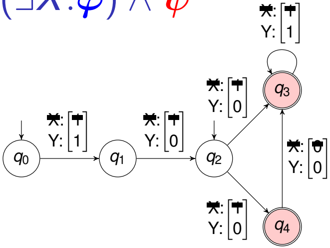


(b) Automaton for ψ

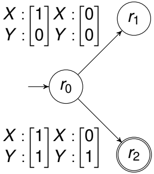
- $\varepsilon \in t_{\exists X.\varphi} \cap t_{\psi} \iff$

Validity checking of $\exists Y.(\exists X.\varphi) \land \psi$





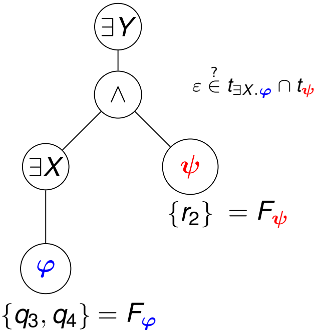
(a) Automaton for $\exists X.\varphi$

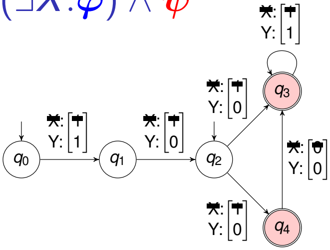


(b) Automaton for ψ

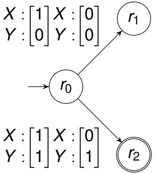
- $\varepsilon \in t_{\exists X.\varphi} \cap t_\psi \iff$

Validity checking of $\exists Y.(\exists X.\varphi) \land \psi$





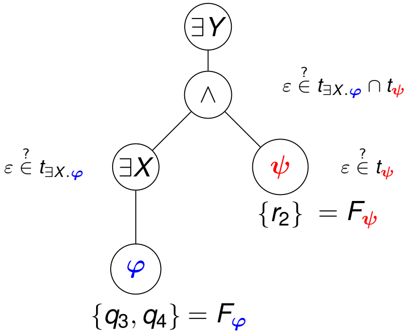
(a) Automaton for $\exists X.\varphi$

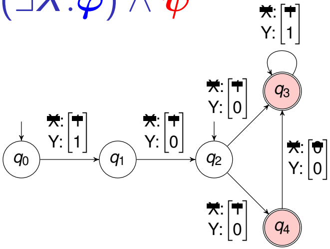


(b) Automaton for ψ

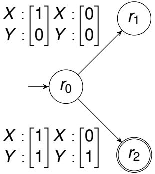
- $\varepsilon \in t_{\exists X.\varphi} \cap t_{\psi} \iff$
  $\iff \varepsilon \in t_{\exists X.\varphi} \land \varepsilon \in t_{\psi}$

Validity checking of $\exists Y.(\exists X.\varphi) \land \psi$





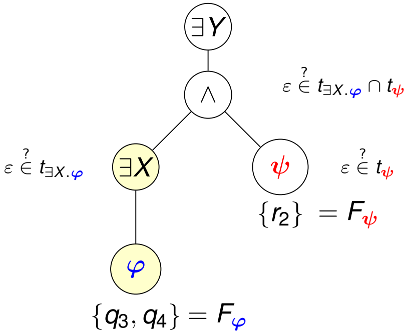
(a) Automaton for $\exists X.\varphi$

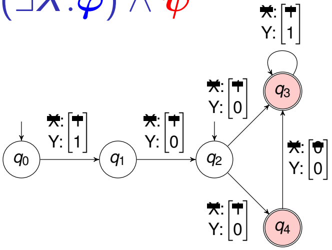


(b) Automaton for ψ

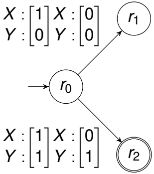
- $\varepsilon \in t_{\exists X.\varphi} \cap t_\psi \iff$
  $\iff \varepsilon \in t_{\exists X.\varphi} \land \varepsilon \in t_\psi$

Validity checking of $\exists Y.(\exists X.\varphi) \wedge \psi$





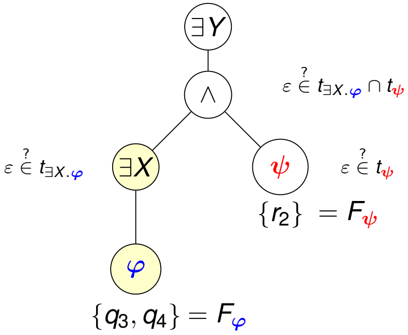
(a) Automaton for $\exists X.\varphi$

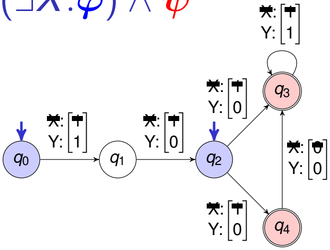


(b) Automaton for ψ

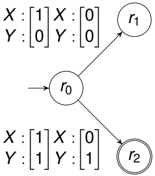
- $\varepsilon \in t_{\exists X.\varphi} \iff \varepsilon \in t_\varphi - \bar{0}^*$
  $\iff \varepsilon \in t_\varphi \vee \varepsilon \in t_\varphi - \bar{0} \vee \varepsilon \in t_\varphi - \bar{0}^2 \dots$
- $\varepsilon \in t_\varphi \iff I_\varphi \cap F_\varphi \neq \emptyset$.

Validity checking of $\exists Y.(\exists X.\varphi) \wedge \psi$





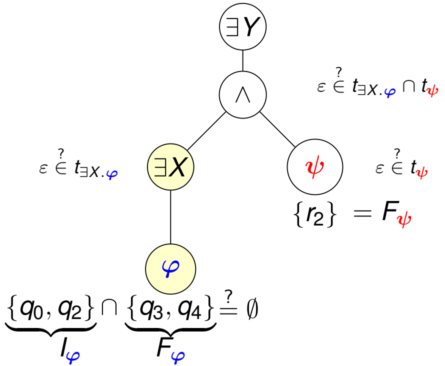
(a) Automaton for $\exists X.\varphi$

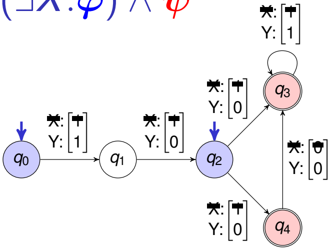


(b) Automaton for ψ

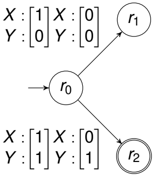
- $\varepsilon \in t_{\exists X.\varphi} \iff \varepsilon \in t_\varphi - \bar{0}^*$
  $\iff \varepsilon \in t_\varphi \vee \varepsilon \in t_\varphi - \bar{0} \vee \varepsilon \in t_\varphi - \bar{0}^2 \dots$
- $\varepsilon \in t_\varphi \iff I_\varphi \cap F_\varphi \neq \emptyset$.

Validity checking of $\exists Y.(\exists X.\varphi) \land \psi$





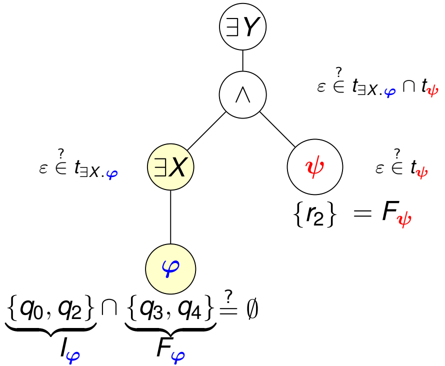
(a) Automaton for $\exists X.\varphi$

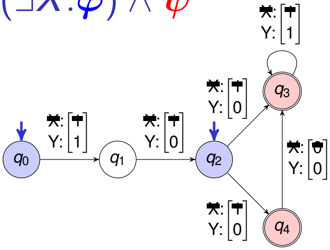


(b) Automaton for ψ

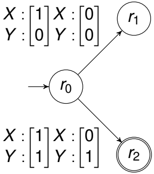
- $\varepsilon \in t_{\exists X.\varphi} \iff \varepsilon \in t_\varphi - \bar{0}^*$
  $\iff \varepsilon \in t_\varphi \lor \varepsilon \in t_\varphi - \bar{0} \lor \varepsilon \in t_\varphi - \bar{0}^2 \ldots$
- $\varepsilon \in t_\varphi \iff I_\varphi \cap F_\varphi \neq \emptyset$.

Validity checking of $\exists Y.(\exists X.\varphi) \land \psi$





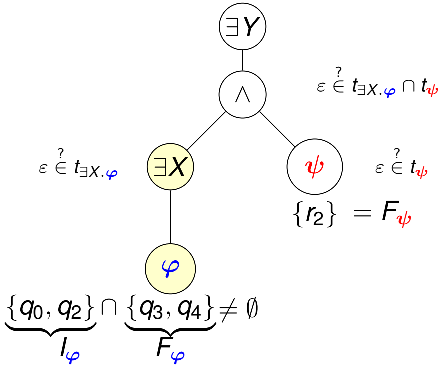
(a) Automaton for $\exists X.\varphi$

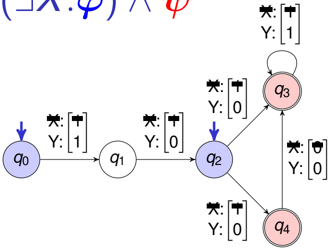


(b) Automaton for ψ

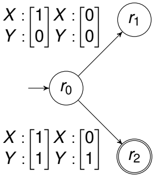
- $\{q_0, q_2\} \cap \{q_3, q_4\} = \emptyset, \ldots$
- ...but we cannot conclude that $\varepsilon \notin t_{\exists X.\varphi}, \ldots$

Validity checking of $\exists Y.(\exists X.\varphi) \land \psi$





(a) Automaton for $\exists X.\varphi$



(b) Automaton for ψ

- $\{q_0, q_2\} \cap \{q_3, q_4\} = \emptyset, \ldots$
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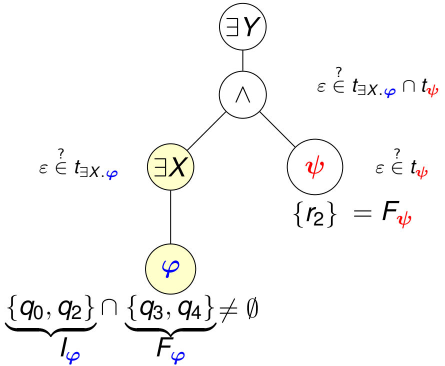
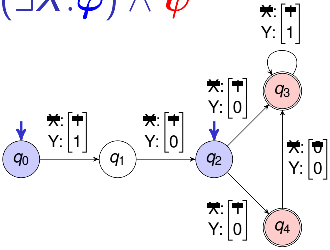
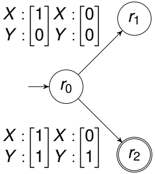
Validity checking of $\exists Y.(\exists X.\varphi) \wedge \psi$





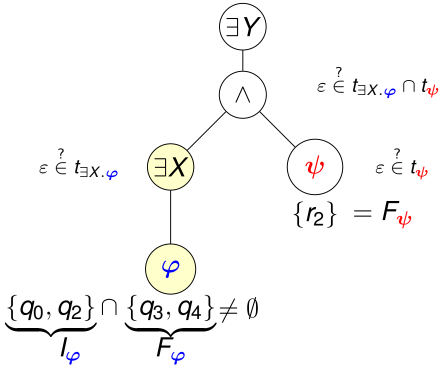
(a) Automaton for $\exists X.\varphi$

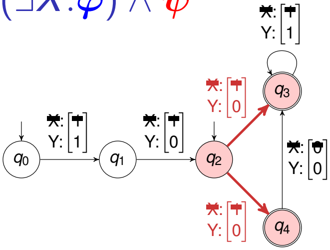


(b) Automaton for ψ

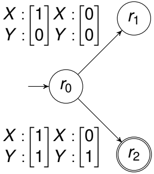
- We have to saturate the final states (because of projection)
- One step of saturation yields set of states $F_{\varphi} - \overline{0}$.

Validity checking of $\exists Y.(\exists X.\varphi) \wedge \psi$





(a) Automaton for $\exists X.\varphi$



(b) Automaton for ψ

- We have to saturate the final states (because of projection)
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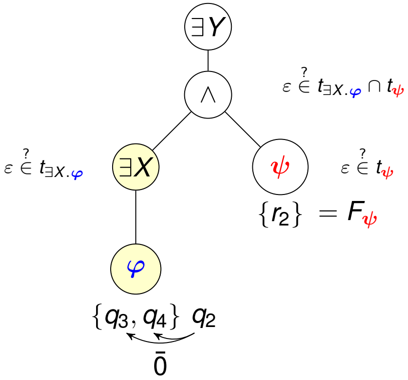
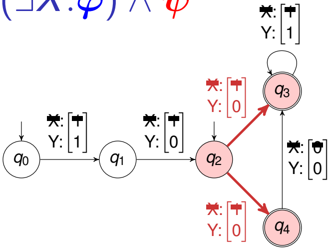
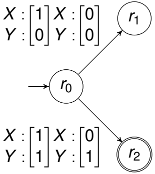
Validity checking of $\exists Y.(\exists X.\varphi) \wedge \psi$





(a) Automaton for $\exists X.\varphi$



(b) Automaton for ψ

- We have to saturate the final states (because of projection)
- One step of saturation yields set of states $F_{\varphi} - \bar{0}$.

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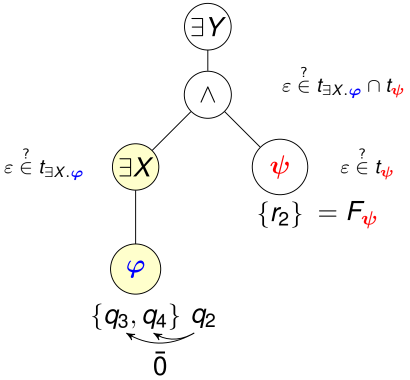

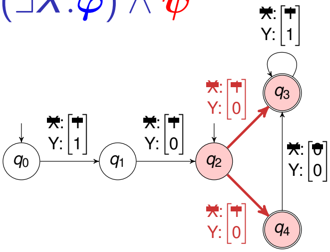

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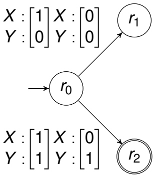

(b) Automaton for ψ

- We repeat the check: $\varepsilon \in t_{\varphi} - \bar{0} \iff$
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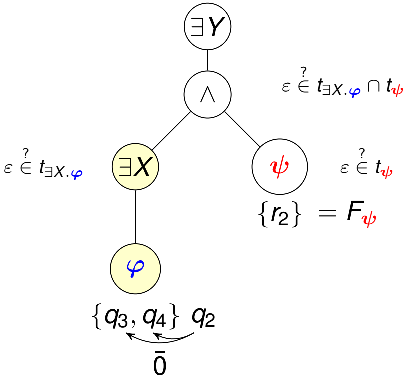

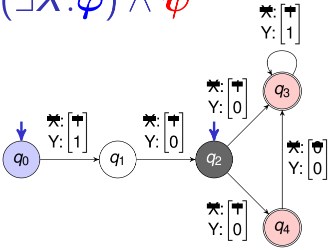

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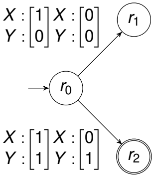

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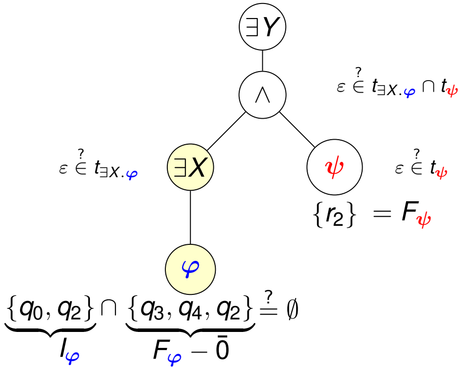

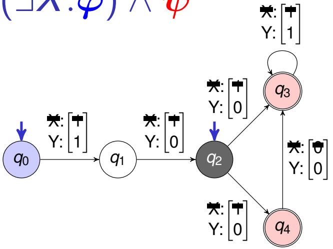

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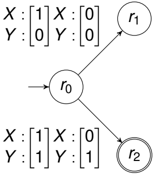

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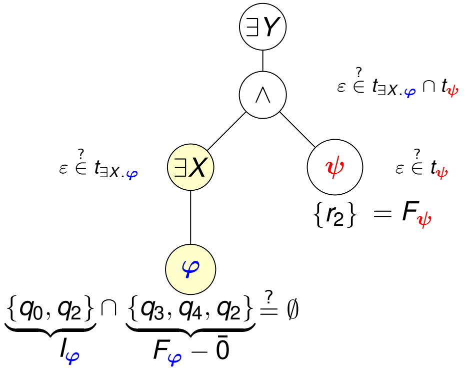

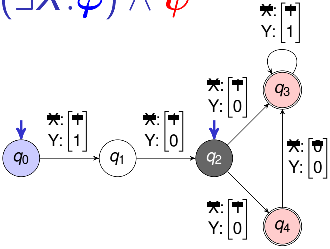

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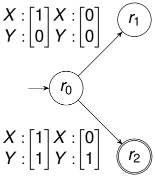

(b) Automaton for ψ

- Since $\{q_0, q_2\} \cap \{q_3, q_4, q_2\} \neq \emptyset$, ...
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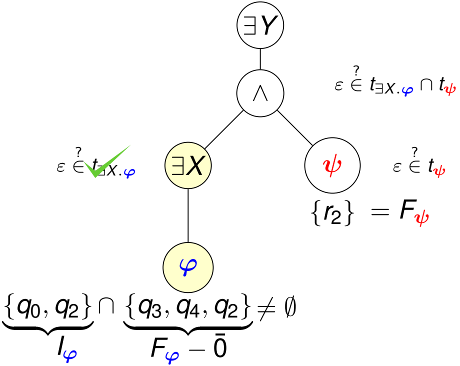

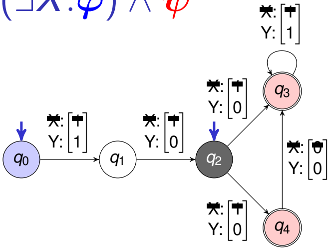

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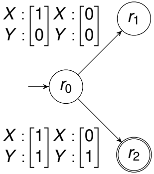

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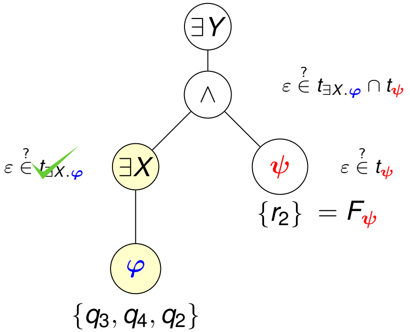

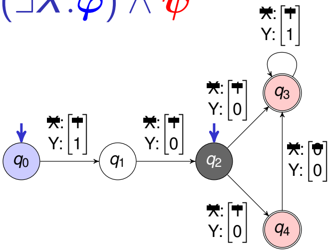

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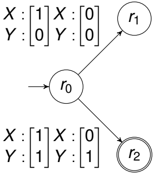

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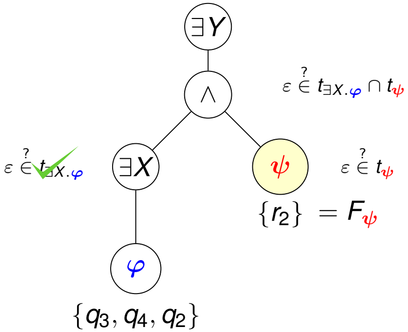

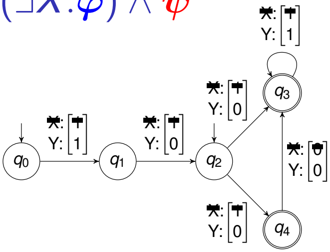

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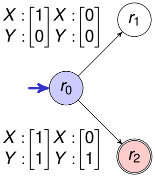

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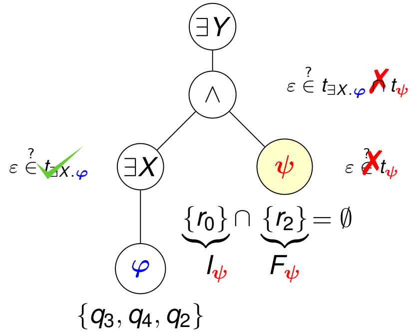

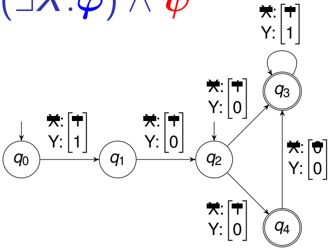

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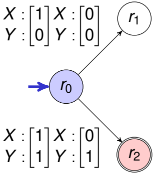

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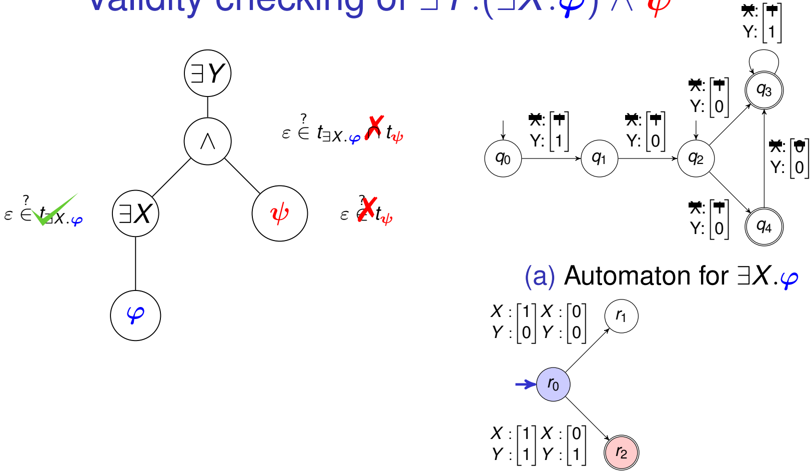

(a) Automaton for $\exists X.\varphi$

(b) Automaton for ψ

- Until we find satisfying member or all of the fixpoints are computed…

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- lazy evaluation
  - if one branch of a binary operator suffices: short-circuit!

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- combination with the explicit automata procedure (MONA)
  - we can prepare a minimal automaton for a subformula
  - reduces the underlying state space
  - various heuristics
    - we explicitly construct quantifier-free subformulae

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  - when computing fixpoints, some elements can subsume others
  - keep fixpoint states minimal (cf. antichains)
  - subsumption even on partially computed elements
- Formula pre-processing
  - pre-processing of the formula can greatly affect performance
  - anti-prenexing — pushing quantifiers down can reduce the explored state space (even exponentially!)

Experimental Evaluation of our tool GASTON

- Results on formulae generated by the UABE tool
  - formulae encode various array invariants
- ∞ represents that the tool timeouted in 2 minutes

Benchmark	MONA		GASTON	
	Time [s]	Space	Time [s]	Space
a-a	**1.51**	30 253	∞	∞
ex10	**6.92**	131 835	11.82	82 236
ex11	4.04	2 393	**0.10**	4 156
ex12	**0.11**	2 591	5.40	68 159
ex13	**0.01**	2 601	0.87	16 883
ex16	**0.01**	3 384	0.18	3 960
ex17	3.15	165 173	**0.09**	3 952
ex18	**0.18**	19 463	∞	∞
ex2	0.10	26 565	**0.01**	1 841
ex20	1.26	1 077	**0.21**	12 266
ex21	**1.51**	30 253	∞	∞
ex4	**0.03**	6 797	0.33	22 442
ex6	**3.69**	27 903	21.44	132 848
ex7	0.75	857	**0.01**	594
ex8	6.83	106 555	**0.01**	1 624
ex9	6.37	586 447	8.31	412 417
fib	**0.04**	8 128	22.15	126 688

Experimental Evaluation of our tool GASTON

- Results on set of parametrized benchmarks up to $k = 20$
- oom(k) represents that the tool run out of memory on formula *k*
- ∞(k) represents that the tool timeouted in 2 minutes on formula *k*

Benchmark	MONA	DWINA	TOSS	COALG	SFA	GASTON
HornLeq	oom(18)	**0.03**	0.08	∞(08)	**0.03**	**0.01**
HornLeq (+3)	oom(18)	∞(11)	0.16	∞(07)	∞(11)	**0.01**
HornLeq (+4)	oom(18)	∞(13)	**0.04**	∞(06)	∞(11)	**0.01**
HornIn	oom(15)	∞(11)	0.07	∞(08)	∞(08)	**0.01**
HornTrans	86.43	∞(14)	N/A	N/A	38.56	**1.06**
SetClosed	oom(05)	∞(14)	∞(03)	∞(01)	∞(04)	∞(06)
SetSingle	oom(04)	∞(08)	0.10	N/A	∞(03)	**0.01**
Ex8	oom(08)	N/A	N/A	N/A	N/A	**0.15**
Ex11(10)	oom(14)	N/A	N/A	N/A	N/A	**1.62**

- DWINA: Fiedor et al.: Nested antichains for WS1S
- TOSS: Ganzow and Kaizer: New algorithm for weak monadic second-order login on inductive structures
- COALG: Traytel: A coalgebraic decision procedure for WS1S
- SFA: D'Antoni and Veanes: Minimization of symbolic automata

Future Work

- extension to WS*k*S
  - weak monadic second-order logic of k successors
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