Nested Antichains for WS1S

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TACAS'15

WS1S

- weak monadic second-order logic of one successor
  - second-order ⇒ quantification over relations;
  - monadic ⇒ relations are unary (i.e. sets);
  - weak ⇒ sets are finite;
  - of one successor ⇒ reasoning about linear structures.
- corresponds to finite automata [Büchi'60]
- decidable — but **NONELEMENTARY**
  - constructive proof via translation to finite automata

Application of WS1S

- allows one to define rich invariants
- famous decision procedure: the MONA tool
  - often efficient (in practice)
- used in tools for checking structural invariants
  - Pointer Assertion Logic Engine (PALE)
  - STRucture ANd Data (STRAND)
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- many other applications
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- but sometimes the complexity strikes back
  - unavoidable in general
  - however, we try to push the usability border further
    - using the recent advancements in non-deterministic automata

WS1S

- Syntax:
  - term ψ ::= $X \subseteq Y \mid \mathrm{Sing}(X) \mid X = \{0\} \mid X = \sigma(Y)$
  - formula φ ::= $\psi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg\varphi \mid \exists X.\varphi$

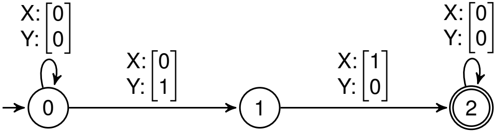
- Interpretation: over finite subsets of $\mathbb{N}$
  - models of formulae = assignments of sets to variables
- sets can be encoded as binary strings:
  - $\{1, 4, 5\} \rightarrow$

Index:	012345		0123456		01234567
Membership:	x✓xx✓✓	,	x✓xx✓✓x	or	x✓xx✓✓xx ...
Encoding:	010011		0100110		01001100

- for each variable we have one track in the alphabet
  - e.g. $\begin{bmatrix}0\\0\end{bmatrix}$ is symbol
- Example: $\{X_1 \mapsto \emptyset, X_2 \mapsto \{4, 2\}\} \models \varphi \overset{def}{\Leftrightarrow} \begin{array}{l} X_1: \\ X_2: \end{array} \begin{bmatrix}0\\0\end{bmatrix} \begin{bmatrix}0\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix} \begin{bmatrix}0\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix} \in L(\mathcal{A}_\varphi)$

Deciding WS1S using deterministic automata

- example of base automaton for $X = \sigma(Y)$



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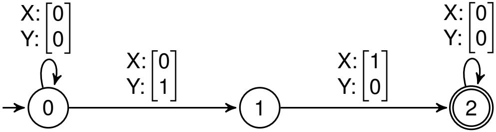

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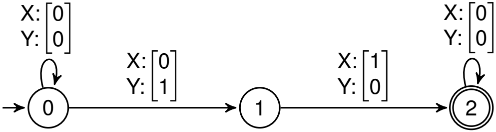

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project W $\longrightarrow \mathcal{A}_4$

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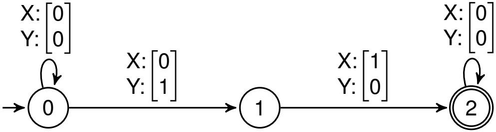

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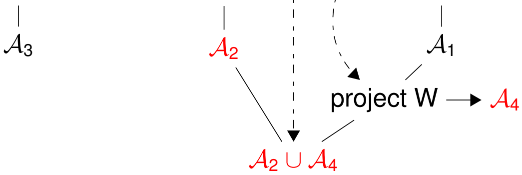

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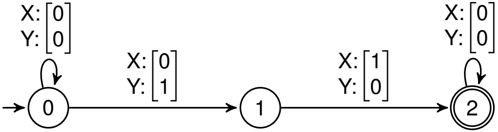

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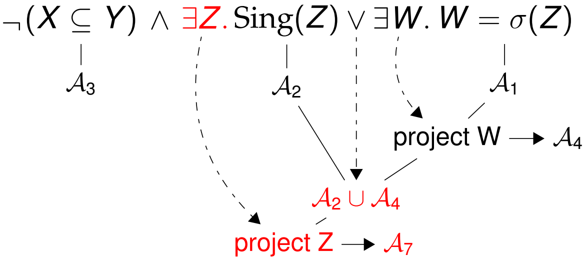

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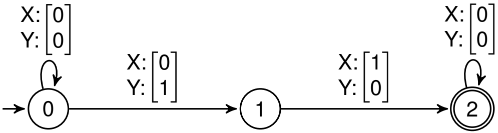

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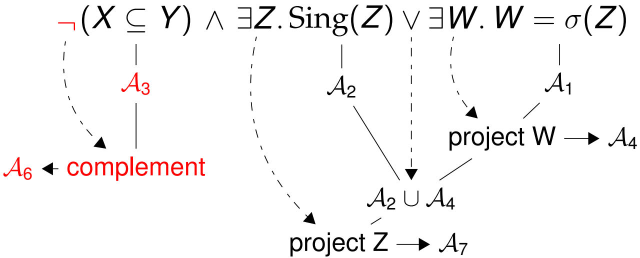

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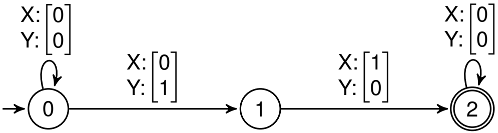

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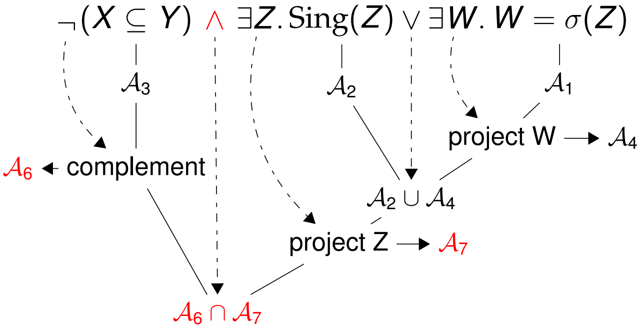
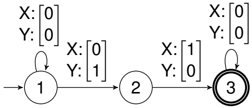


How to handle quantification

- issue with projection (existential quantification)
  - after removing of the tracks not all models would be accepted
  - so we need to adjust the final states



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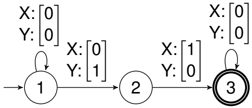

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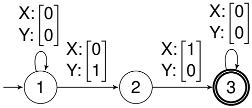

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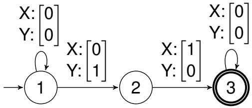

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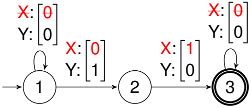

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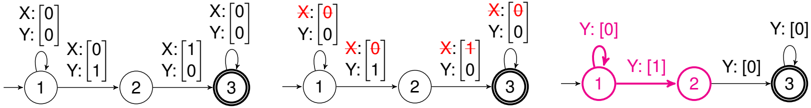

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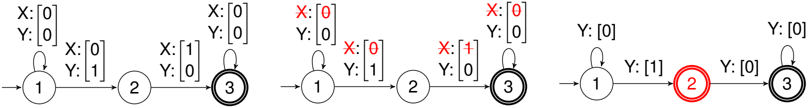

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Deciding WS1S using non-deterministic automata

- we consider only formulae in Prenex Normal Form (∃PNF)
  - we focus on dealing with prefix and alternations of quantifications
- based on number of alternations m

$$\underbrace{\varphi = \neg\,\exists \mathcal{X}_m \neg \ldots \neg\,\exists \mathcal{X}_2 \underbrace{\neg\,\exists \mathcal{X}_1 : \varphi_0(\mathbb{X})}_{\varphi_1}}_{\varphi_m} \tag{1}$$

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→ hierarchical family of automata defined as follows:

- $\mathcal{A}_{\varphi_0}$ = by composition of atomic automata (previously described)
- $\mathcal{A}_{\varphi_m} = (\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^{Q_0}}}}}}}_{m}, \Delta_m, I_m, F_m)$

The intuition behind the procedure

Key observation for ground formulae

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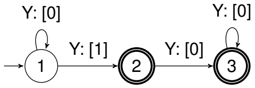

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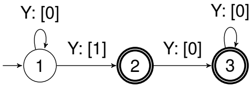

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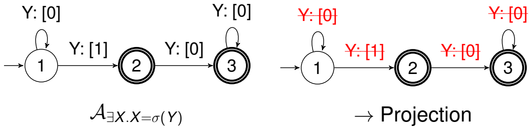

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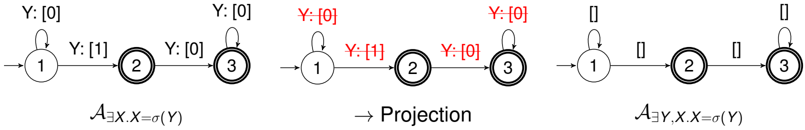

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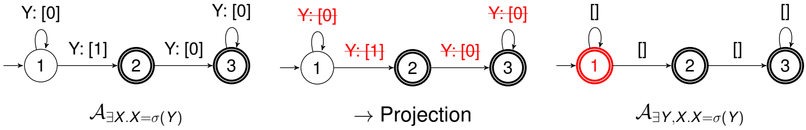

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- final states are more tricky
  - issue with projection (previously described)
  - multiple levels of determinisation

Introduction to the computation of final states

- we already have:
  - formula in ∃PNF: $\varphi = \neg \exists \mathcal{X}_m \neg \ldots \neg \exists \mathcal{X}_2 \neg \exists \mathcal{X}_1 : \varphi_0(\mathbb{X})$
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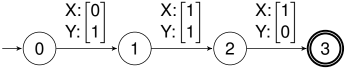
- our proposed method
  - is based on generalized backward reachability of final states
  - works on symbolic representation of states, sets of states, sets of sets of states …
    - for non-final states $\rightarrow$ compute their controllable predecessors $cpre_0$
      (Intuition) states that lead outside of non-final states become final after negation
    - for final states pre_0 $\rightarrow$ compute their predecessors
      (Intuition) states that lead to final states become non-final after negation
  - prunes states on all levels of the hierarchy to achieve minimal representation

Towards symbolic representation

- Motivating example: $\neg\exists X.\varphi$
  - $Q = \{0, 1, 2, 3\}$
  - $F = \{3\}$



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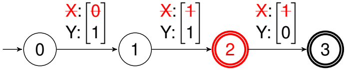

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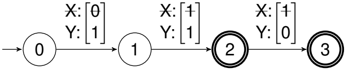

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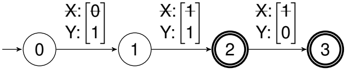

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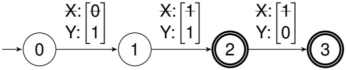

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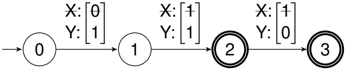

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- so why not work with this symbolic representation only?

Computing final states F_m of formula φ_m

- Given $\varphi = \neg \exists \mathcal{X}_m \neg \ldots \neg \exists \mathcal{X}_2 \neg \exists \mathcal{X}_1 : \varphi_0(\mathbb{X})$

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3. Reduce set of non-final states after projection:
   $N_1^{\exists} = \{\nu Z . N_i \cap \mathtt{cpre}_0(Z)\}$
   - Notice the duality with step 1.

$$\cap \;\mapsto\; \cup \qquad \mathtt{cpre}_0 \;\mapsto\; \mathtt{pre}_0 \qquad \nu \;\mapsto\; \mu \tag{2}$$

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4. Negate the non-final states: $F_2 =\downarrow \{N_1^{\exists}\}$

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⋮

5. and keep alternating between computing final and non-final states until F_m as follows:
   - $F_{i+1} = \downarrow \{\nu Z . N_i \cap \mathtt{cpre}_0(Z)\}$
   - $N_{i+1} = \uparrow \{\mu Z . F_i \cup \mathtt{pre}_0(Z)\}$

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- Can we compute $cpre_0/pre_0$ of symbolic states?

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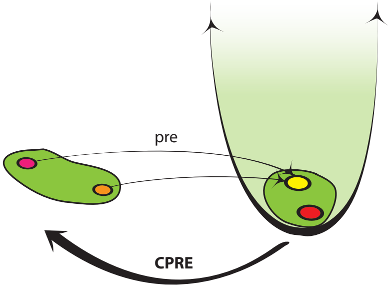

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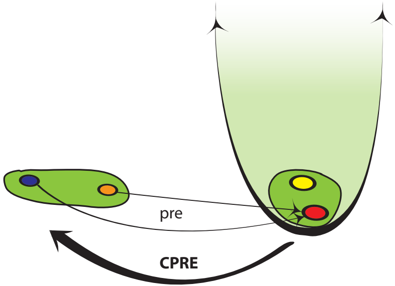

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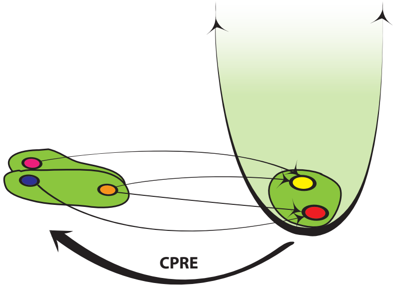

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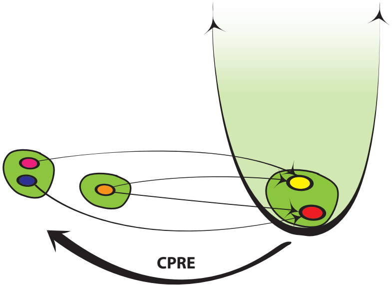

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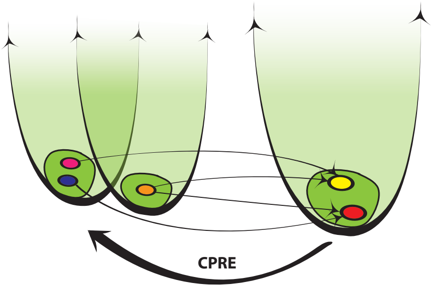

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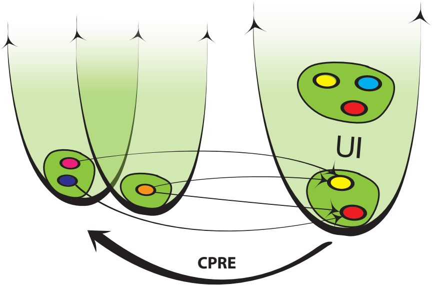

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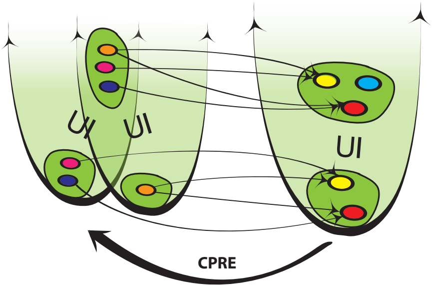

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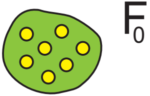
A summary of the inner structure of F_m



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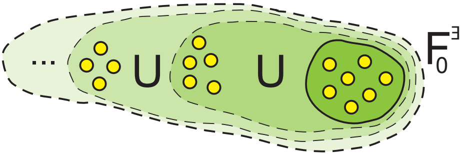

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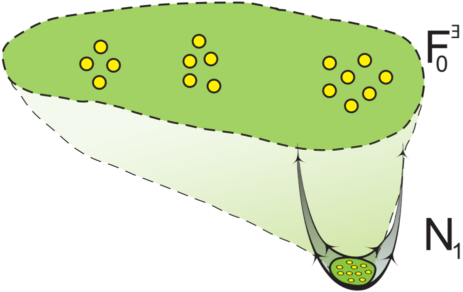

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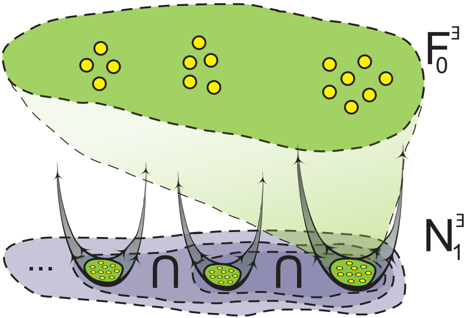

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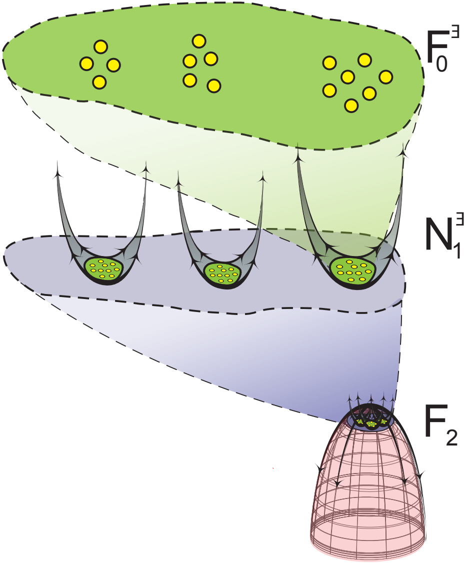

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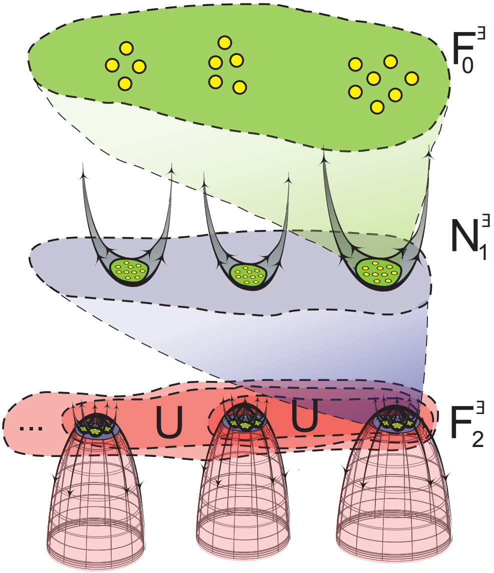

How to achieve state space reduction

- We showed the nested structure of F_m is very complex,
  - but we only work with the symbolic representation of the generators (with antichains)
  - . . . and the generators of the generators and . . .
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  - ...and the generators of the generators and ...
  - this itself is the first source of space reduction
- further we prune the generators subsumed by other generators
  - the subsumption relation is computed on nested structure of symbolic representation of lower levels

Experimental results

- implemented in dWiNA
- compared with MONA:
  - on generated and real formulae
  - in generic and ∃PNF form

	MONA				dWiNA	
	Time [s]		Space [states]		Time [s]	Space [states]
real	normal	∃PNF	normal	∃PNF	Prefix	Prefix
list-reverse-after-loop	0.01	0.01	179	1 326	0.01	100
list-reverse-in-loop	0.02	0.47	1 311	70 278	0.02	260
bubblesort-else	0.01	0.45	1 285	12 071	0.01	14
bubblesort-if-else	0.02	2.17	4 260	116 760	0.23	234
bubblesort-if-if	0.12	5.29	8 390	233 372	1.14	28
generated						
3 alternations	-	0.57	-	60 924	0.01	50
4 alternations	-	1.79	-	145 765	0.02	58
5 alternations	-	4.98	-	349 314	0.02	70
6 alternations	-	**TO**	-	**TO**	0.47	90

Future Work

- extension to WS2S
  - opens whole new world of tree structures
  - implementation with tree automata
- generalization of symbolic tree representation
  - to process logical connectives
  - to handle general (non-$\exists$PNF) formulae
- syntactical optimizations
  - using Direct Acyclic Graph (DAG) for representation of formulae
  - anti-prenexing
  - smarter conversion to $\exists$PNF

Thank you for your attention!

Any questions?