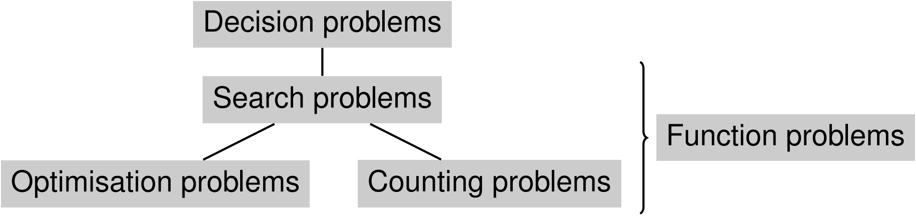
Counting Classes

Complexity Theory

Classification of Problems



Given a relation $R \subseteq X \times Y$ and $x \in X$:

- Decision problems: decide membership in a language (yes/no).
  - Is there some $y \in Y$ s.t. $R(x, y)$?
- Function problems: generate some additional output.
  - **Search problems**: Find any $y \in Y$ s.t. $R(x, y)$.
  - **Optimisation problems**: Find the best $y \in Y$ s.t. $R(x, y)$.
  - **Counting problems**: How many $y \in Y$ are there s.t. $R(x, y)$?

Counting Problems

Definition (Counting problem)

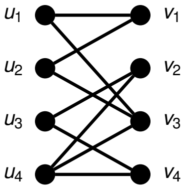
Consider a relation $R \subseteq X \times Y$ and the decision problem $D_R \subseteq X$ s.t. $x \in D_R \iff \exists y \in Y \,.\, R(x, y)$. The counting problem associated with R, $\#D_R$, is defined as

$$\#D_R(x) = |\{y \in Y \mid R(x, y)\}| \ .$$

Examples:

- #SAT: how many different assignments satisfy given formula?
- #CLIQUE: how many cliques of size k or larger are in a graph?
- #HAMILTONIAN PATH: how many different Hamiltonian paths are in a graph?

Example: Counting Perfect Matchings



Definition (MATCHING)

Is there a perfect matching in the bipartite graph $G = (U, V, E)$?

Definition (#MATCHING)

How many perfect matchings are in the bipartite graph $G = (U, V, E)$?

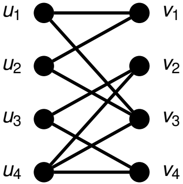
Example: Counting Perfect Matchings

Recall that MATCHING can be solved by checking whether the determinant of the adjacency matrix $\boldsymbol{A}^{\boldsymbol{G}}$ of G is not identically zero.

$$\det \boldsymbol{A}^{\boldsymbol{G}} = \sum_{\pi} \left(\sigma(\pi) \prod_{i=1}^{n} \boldsymbol{A}^{\boldsymbol{G}}_{i,\pi(i)} \right)$$

where

- π ranges over all permutation of n elements,
- $\sigma(\pi) = 1$ if π contains an even number of transpositions, else -1.



$$\boldsymbol{A}^{\boldsymbol{G}} = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline u_1 & 1 & 0 & 1 & 0 \\ u_2 & 1 & 0 & 1 & 0 \\ u_3 & 0 & 1 & 0 & 1 \\ u_4 & 0 & 1 & 1 & 1 \end{array}$$

Example: Counting Perfect Matchings

$$\det \boldsymbol{A^G} = \sum_{\pi} \left(\sigma(\pi) \prod_{i=1}^{n} \boldsymbol{A}^{\boldsymbol{G}}_{i,\pi(i)} \right)$$

- Note that the summation is done over all perfect matchings, but including the undesirable $\sigma(\pi)$ element.

Example: Counting Perfect Matchings

$$\det \boldsymbol{A}^{\boldsymbol{G}} = \sum_{\pi} \left(\sigma(\pi) \prod_{i=1}^{n} \boldsymbol{A}^{\boldsymbol{G}}_{i,\pi(i)} \right)$$

- Note that the summation is done over all perfect matchings, but including the undesirable $\sigma(\pi)$ element.
- If we get rid of the $\sigma(\pi)$ element, we arrive at a different characteristic of a matrix called the permanent.

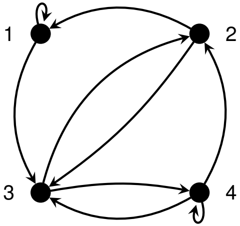
$$\operatorname{perm} \boldsymbol{A}^{\boldsymbol{G}} = \sum_{\pi} \left(\prod_{i=1}^{n} \boldsymbol{A}^{\boldsymbol{G}}_{i,\pi(i)} \right)$$

- The permanent of $\boldsymbol{A}^{\boldsymbol{G}}$ is precisely the number of perfect matchings in G, the problem is therefore known as PERMANENT.

Example: Counting Perfect Matchings

- Further, the number of perfect matchings in $G = (U, V, E)$ is equal to the number of cycle covers in the directed graph

$$G' = (\{1, \ldots, |U|\}, \{(i,j) \mid (u_i, v_j) \in E\}) \ .$$



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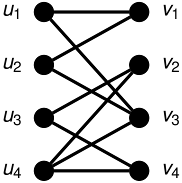

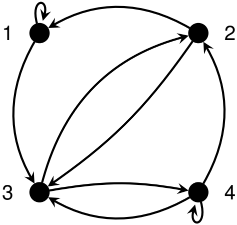

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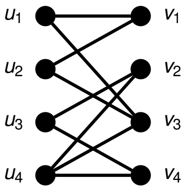

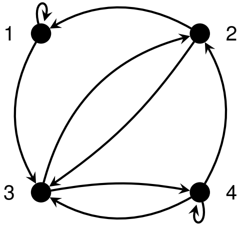

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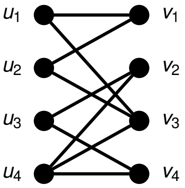
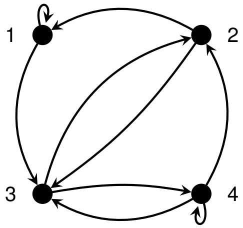
$$\operatorname{perm} \boldsymbol{A}^{\boldsymbol{G}} = \sum_{\pi} \left(\prod_{i=1}^{n} \boldsymbol{A}^{\boldsymbol{G}}_{i,\pi(i)} \right) = \ ?$$

Example: Counting Perfect Matchings





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$$\operatorname{perm} \boldsymbol{A}^{\boldsymbol{G}} = \sum_{\pi} \left(\prod_{i=1}^{n} \boldsymbol{A}^{\boldsymbol{G}}_{i,\pi(i)} \right) = 4$$

Example: Graph Reliability

Counting is relevant to probability; consider the decision problem

Definition (REACHABILITY)

Given a graph G, is there a path from node u to node v?

This gives rise to the following counting problem:

Definition (GRAPH RELIABILITY)

Given a graph G with m edges, how many of the 2^m subgraphs of G contain a path from node u to node v?

The problem is called GRAPH RELIABILITY because it gives a precise estimate of the probability that u and v will remain connected when all edges fail independently with probability $\frac{1}{2}$ each.

$\#$**P**

Definition ($\#$**P**)

$\#$**P** is the class of all counting problems associated with polynomially balanced polynomial-time decidable relations.

- $\#$**P** is pronounced “number **P**”, “sharp **P**”, or “pound **P**”.
- Polynomially balanced relation: if $R(x, y)$, then $|y| \leq p(|x|)$.
- Polynomial-time decidable relation:
    - given x and y, it is checkable in polynomial time whether $R(x, y)$.

Reduction of Counting Problems

- All decision problems are easily reducible to their corresponding counting problems.

- As with other function problems, a reduction between counting problems A and B consists of two parts:
  - part R mapping instances x of A to instances $R(x)$ of B,
  - part S recovering from the answer y of $R(x)$ the answer $S(y)$ of x.

- For counting problems, there is a convenient class of reductions:

Definition (Parsimonious Reduction)

A reduction is parsimonious when $S = \text{id}$.

$\#$SAT is $\#$**P**-complete

Theorem

$\#$SAT *is* $\#$**P***-complete.*

Proof.

Parsimonious variant of Cook's theorem (for CIRCUIT SAT):

- Each polynomially balanced and polynomial-time decidable binary relation $R \subseteq X \times Y$ together with $x \in X$ can be in deterministic polynomial time reduced to a CNF formula $\phi_{R(x)}$ with *input variables* $I = \{i_1, \ldots, i_n\}$.
- Each satisfying truth assignment to I corresponds to a unique $y \in Y$ s.t. $R(x, y)$. □

PERMANENT is $\#\mathbf{P}$-complete.

Theorem (Valiant's Theorem)

PERMANENT *is* $\#\mathbf{P}$*-complete.*

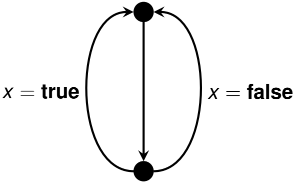
- Interesting because MATCHING $\in \mathbf{P}$.

Proof. (idea)

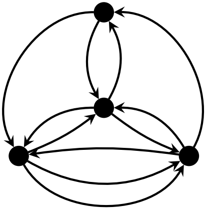
- By reduction from $\#$SAT.
- For a 3SAT formula ϕ, we construct a graph G_ϕ such that the cycle covers of G_ϕ somehow correspond to satisfying assignments of ϕ.
- The construction is very similar to the proof of **NP**-completeness of HAMILTONIAN PATH.

PERMANENT is $\#\mathbf{P}$-complete.

- For each Boolean variable x in ϕ, we create a choice gadget.
- For each clause in ϕ, we create a clause gadget:
  - no cycle cover traverses all 3 external edges,
  - for any proper subset S of external edges (including $\emptyset$), there is exactly one cycle cover traversing only external edges from S and no other external edges.



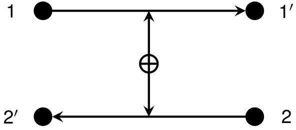
choice gadget



clause gadget

PERMANENT is $\#\mathbf{P}$-complete.

- External edges from clause gadgets are connected to corresponding edges of choice gadgets using XOR gadgets:
  - if exactly one of the edges $(1, 1')$ or $(2, 2')$ is traversed, the number of cycle covers is multiplied by 4,
  - there is no cycle cover in the graph if none or both are traversed.



XOR gadget

- For each satisfying assignment of ϕ, there are 4^m cycle covers
  - where m is the total number of literal occurrences in the formula.

- Details are rather technical and can be found in the literature:
  - structure of the XOR gadget,
  - reduction to PERMANENT MOD N.

□

How Strong Is Counting?

- Counting is very powerful indeed!
- Is $\#$**P** more powerful than **PH**?

How Strong Is Counting?

- Counting is very powerful indeed!
- Is $\#$**P** more powerful than **PH**?
- Note that we cannot directly compare $\#$**P** to **PH**:
  - $\#$**P** . . . a class of functions,
  - **PH** . . . a class of languages.
- However, recall the class **PP**:
  - **PP** . . . the class of languages L s.t. there is a poly. nondet. TM M, $x \in L$ iff more than $\frac{1}{2}$ computations of M on x end up accepting.
- There is a close relation between $\#$**P** and **PP**:
  - try looking at the MSB of the number of accepting computations.

Theorem (Toda's Theorem)

$$\mathbf{PH} \subseteq \mathbf{P}^{\mathbf{PP}}$$

$\oplus$**P**

Definition ($\oplus$**P**)

$\oplus$**P** is the class of languages L for which there is a polynomially balanced polynomial-time decidable relation R such that $x \in L$ iff the number of y's such that $R(x, y)$ is odd.

- $\oplus$**P** is pronounced "odd **P**", or "parity **P**".
- $\oplus$SAT and $\oplus$HAMILTONIAN PATH are $\oplus$**P**-complete,
  - a reduction similar to $\#$SAT and $\#$HAMILTONIAN PATH.

$\oplus\mathbf{P} = \mathbf{co}\oplus\mathbf{P}$

Theorem

$\oplus\mathbf{P}$ *is closed under complement, i.e.*

$$\oplus\mathbf{P} = \mathbf{co}\oplus\mathbf{P}\ .$$

Proof.

- The complement of $\oplus$SAT is obviously **co$\oplus$P**-complete.
- This language reduces to $\oplus$SAT of $\phi(x_1, \ldots, x_n)$ as follows:
  1. Add a new variable z to each clause of ϕ.
  2. Also add n clauses $(z \implies x_i)$ for $1 \leq i \leq n$.
- Any SAT assignment in the old formula is still SAT ($z =$ **false**).
- We get a new all-**true** SAT assignment ($z =$ **true**). □

$\mathbf{NP} \subseteq \mathbf{RP}^{\oplus\mathbf{P}}$

Theorem

$$\mathbf{NP} \subseteq \mathbf{RP}^{\oplus\mathbf{P}}$$

- **RP** . . . the class of languages for which there exists a polynomial Monte Carlo Turing machine.

Proof. (idea)

- Construct a polynomial MC TM for SAT using an oracle for $\oplus$SAT.
- We are given formula ϕ over variables $\{x_1, \ldots, x_n\}$.
- For $S \subseteq \{x_1, \ldots, x_n\}$ a hyperplane η_S is a Boolean expression in CNF stating an even number among the variables in S are **true**.
  - For variables $y_0, \ldots, y_n$, η_S is the conjunction of clauses (y_0), (y_n),

$$\text{plus for each } 1 \le i \le n \quad \begin{cases} (y_i \iff (y_{i-1} \oplus x_i)) & \text{if } x_i \in S \\ (y_i \iff y_{i-1}) & \text{if } x_i \notin S \end{cases}$$

$\mathbf{NP} \subseteq \mathbf{RP}^{\oplus\mathbf{P}}$

- The algorithm:

```
1 $\phi_0 := \phi$
2 For $i = 1, \ldots, n+1$ repeat the following:
    1 Generate a random subset $S_i \subseteq \{x_1, \ldots, x_n\}$.
    2 Set $\phi_i = \phi_{i-1} \wedge \eta_{S_i}$.
    3 If $\phi_i \in \oplus$SAT answer "$\phi$ is satisfiable."
    4 Else continue.
3 Answer "$\phi$ is probably unsatisfiable."
```

- The probability of a false negative is no larger than $\frac{7}{8}$.
  - becomes less than $\frac{1}{2}$ by repeating the algorithm 6×. □