Negated String Containment is Decidable

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Negated string containment
and its semantics

NEGATED STRING CONTAINMENT

Input: A formula

$$\varphi \triangleq \neg \mathit{contains}(\mathcal{N}, \mathcal{H}) \land \bigwedge_{X \in \mathbb{X}} X \in L_X$$

with $\mathcal{N}, \mathcal{H} \in (\Sigma \cup \mathbb{X})^*$ and L_X is regular for every variable X.

Question: Find a morphism $\sigma : \mathbb{X} \to \Sigma^*$ such that

- $\sigma(X) \in L_X$ for every variable X, and
- $\sigma(\mathcal{N})$ is not a factor $\sigma(\mathcal{H})$.

We call $\mathcal{N}$ and $\mathcal{H}$ the $\mathcal{N}$eedle and $\mathcal{H}$aystack, respectively.

Negated string containment

Examples

Given

$$\varphi \triangleq \neg contains(XabY, YababX) \quad \wedge \quad X \in a^* \quad \wedge \quad Y \in b^*$$

We have

- $\{X \mapsto a, Y \mapsto b\} \models \varphi$
- $\{X \mapsto \varepsilon, Y \mapsto \varepsilon\} \not\models \varphi$

Negated string containment
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- $\{X \mapsto a, Y \mapsto b\} \models \varphi$
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Alternatively,

$$\varphi' \triangleq \neg contains(XabY, YababX) \quad \wedge \quad X \in (ab)^* \quad \wedge \quad Y \in (ab)^*$$

has no models.

Motivation

Symbolic execution and SMT solving

```
# Check if pwd can be writen as concat(w, ..., w) for some word w
pwd = input()  # pwd='abab'
pwd2 = concat(pwd, pwd) # pwd2='abababab'
pwd2_inner = pwd2[1:-1] # pwd2_inner='bababa'
if is_substring(pwd, pwd2_inner):
    report_weak_password()
else:
    proceed()
```

What value of `pwd` causes `proceed()` to be called?

$$P_2 = P_1 \circ P_1 \quad \wedge \quad P_2 = U \circ P_3 \circ V \wedge U, V \in \Sigma \quad \wedge \quad \neg contains(P_1, P_3)$$

Challenge

The $\neg contains(\mathcal{N}, \mathcal{H})$ formula can be equivalently expressed as an $\exists\forall$-quantified disequation

$$\exists \vec{X}(\neg contains(\mathcal{N}, \mathcal{H})) \quad \Leftrightarrow \quad \exists \vec{X}\, \forall P, S(P \circ \mathcal{N} \circ S \neq \mathcal{H})$$

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Quantifiers are notoriously difficult, quickly leading to undecidability

- already the $\exists^1\forall^1\exists^3$-fragment is known to be undecidable[1]

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Preliminaries

Connection between automata and Presburger arithmetic

Theorem (Modified Parikh theorem)

Let $\mathcal{A}$ be an NFA. There is an effectively constructable Presburger arithmetic (PA) formula φ_{Parikh} of size polynomial in $|\mathcal{A}|$ such that

1. *any model $\sigma \models \varphi_{Parikh}$ corresponds to an accepting run ρ of $\mathcal{A}$, and*
2. *$\sigma(q \xrightarrow{a} r)$ is the number of times the transition $q \xrightarrow{a} r$ is taken by ρ.*

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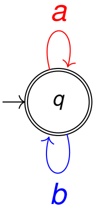
We can reason about automaton runs in decidable Presburger arithmetic. However, commutativity prevents precise reasoning.



$$\sigma = \{q \xrightarrow{a} q \mapsto 1, q \xrightarrow{b} q \mapsto 1\}$$

$$\sigma \models \varphi_{Parikh}$$

$$\sigma \rightsquigarrow w_1 = ab \in L$$

$$\sigma \rightsquigarrow w_2 = ba \in L$$

Preliminaries

Flat languages

Regular language L is flat if it has the form:

$$L = \bigcup_{1 \leq i \leq N} u_{i,0}(w_{i,1})^* \cdots (w_{i,k_i})^* u_{i,k_i}$$

where $u_{i,j}, w_{i,k} \in \Sigma^*$.

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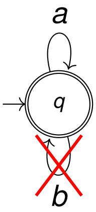
A flat language is a regular language for which every model of its φ_{Parikh} corresponds to exactly one $w \in L$.



Preliminaries

Decision procedure for flat ¬*contains*

If all variables are flat, we can reason precisely about variable assignments in PA, i.e., we can construct an equisatisfiable quantified PA formula.

Theorem

Let φ be a formula

$$\varphi \triangleq \neg \mathit{contains}(\mathcal{N}, \mathcal{H}) \wedge \bigwedge_{X \in \mathbb{X}} X \in L_X$$

such that, for every variable X, the language L_X is flat. Then satisfiability of φ is decidable[a]*.*

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[a] Yu-Fang Chen et al. “A Uniform Framework for Handling Position Constraints in String Solving”. In: PLDI (2025).

Narrowing down the question

When is $\neg contains(\mathcal{N}, \mathcal{H})$ easy?

$$\varphi = \neg contains(\mathcal{N}, \mathcal{H}) \wedge \bigwedge_{X \in \mathbb{X}} X \in L_X$$

Solving φ is **easy** when:

- we can find $\sigma : \mathbb{X} \to \Sigma^*$ such that $|\sigma(\mathcal{N})| > |\sigma(\mathcal{H})|$ (using φ_{Parikh}),
- all variables are flat, or
- $\mathcal{N}$ is a literal.

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Solving φ is **hard** when every non-flat variable $X \in \mathbb{X}$ satisfies:

1. X occurs both in $\mathcal{H}$ and $\mathcal{N}$, or
2. X occurs only in $\mathcal{H}$.

Step 1: Normalization

Restricting the structure of regular languages

Step 1: Normalization

Input: A formula

$$\varphi \triangleq \neg contains(\mathcal{N}, \mathcal{H}) \land \bigwedge_{X \in \mathbb{X}} X \in L_X$$

Output: An equisatisfiable disjunction

$$\bigvee_{i \in I} (\neg contains(\mathcal{N}_i, \mathcal{H}_i) \land \bigvee_{X \in \mathbb{X}} X \in L_{X,i})$$

such that

- every flat variable X has a language w_X^* for some $w_X \in \Sigma^*$,
- every non-flat variable Y has a language S_Y^* for some $S_Y \subseteq \Sigma^*$.

$$\neg \mathit{contains}(\mathcal{N}, \mathcal{H}) \land \bigwedge_{X \in \mathbb{X}} X \in L_X$$

How to handle non-flat variables occurring both in $\mathcal{H}$ and $\mathcal{N}$?

Dealing with situations when Y is on both sides

Our goal

Ultimately, we obtain the following lemma.

Lemma

Let $Y \in \mathbb{X}$ be a non-flat variable, $\square \notin \Sigma$ be a fresh symbol and

$$\varphi = \neg contains(u_0 Y u_1 \cdots Y u_n, v_0 Y v_1 \cdots Y v_m) \wedge \bigwedge_{X \in \mathbb{X}} X \in L_X.$$

Then φ is equisatisfiable to φ', where

$$\varphi' = \neg contains(u_0 \square u_1 \cdots \square u_n, v_0 \square v_1 \cdots \square v_m) \wedge \bigwedge_{X \in \mathbb{X} \setminus \{Y\}} X \in L_X.$$

Dealing with situations when Y is on both sides

Proof sketch, direction from φ' to φ

Let

$$\varphi = \neg contains(u_0 Y u_1 \cdots Y u_n, v_0 Y v_1 \cdots Y v_m) \wedge \bigwedge_{X \in \mathbb{X}} X \in L_X$$

$$\varphi' = \neg contains(\underbrace{u_0 \square u_1 \cdots \square u_n}_{\mathcal{N}'}, \underbrace{v_0 \square v_1 \cdots \square v_m}_{\mathcal{H}'}) \wedge \bigwedge_{X \in \mathbb{X} \setminus \{Y\}} X \in L_X.$$

Assume that we have an assignment $\sigma' : \mathbb{X} \setminus \{Y\} \to \Sigma^*$, such that $\sigma' \models \varphi'$.

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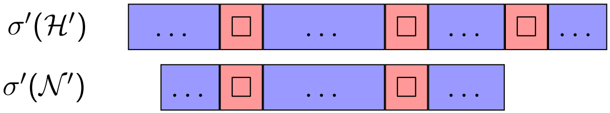
Intuitively, σ' is interesting only when $\square$ in $\sigma'(\mathcal{N}')$ is above some $\square$ in $\sigma'(\mathcal{H}')$.



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Let $w_Y \in L_Y$, and let $\sigma \triangleq \sigma' \triangleleft \{ Y \mapsto w_Y \}$.

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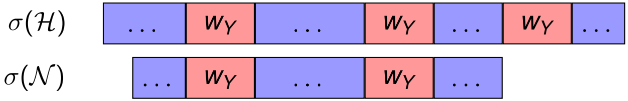
Since $\sigma' \models \neg contains(\mathcal{N}', \mathcal{H}')$ we know that



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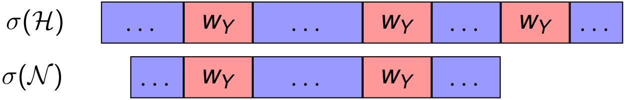
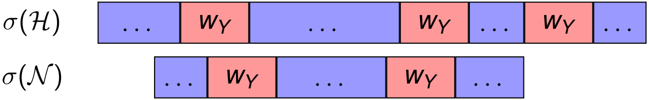


contains a conflict.

Therefore, if $\sigma \not\models \neg \mathit{contains}(\mathcal{N}, \mathcal{H})$, we cannot have every w_Y in $\sigma(\mathcal{N})$ under some w_Y in $\sigma(\mathcal{H})$.

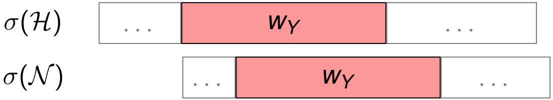


Dealing with situations when Y is on both sides

Proof sketch, direction from φ' to φ

If $\sigma \not\models \neg \mathit{contains}(\mathcal{N}, \mathcal{H})$, we can force some w_Y from $\sigma(\mathcal{N})$ to overlap with w_Y from $\sigma(\mathcal{H})$

- by picking long enough w_Y



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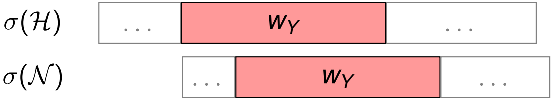


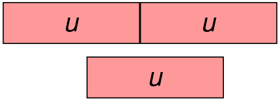
All that is needed is to come up with a special w_Y that cannot have conflict-free overlaps (of sufficient size) with itself, which would allow us to *always* construct a model σ from σ'.

Enter combinatorics on words

How to choose w_Y with the desired properties

A word u is called *primitive* if $u \notin w^*$ for any word $w \neq u$.

Primitive words have cool properties, e.g., if $uu = pus$, then either $p = \varepsilon$ or $s = \varepsilon$.
Graphically, the following is not possible.



Applying combinatorics on words

Proof sketch, direction from φ' to φ

Thanks to our normalization, we have $\{u, v\}^* \subseteq L_Y$ with $u, v \notin w^*$ for any word w.

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Let us define α and β as

$$\alpha \triangleq u^2 u^k v^2 \quad \in L_Y$$
$$\beta \triangleq u^2 v^l v^2 \quad \in L_Y$$

for $k = \mathrm{lcm}(|u|, |v|)/|u|$ and $l = \mathrm{lcm}(|u|, |v|)/|v|$.

Lemma

Both α and β are primitive.[a]

[a] R. C. Lyndon and M. P. Schützenberger. "The equation $a^M = b^N c^P$ in a free group.". In: *Michigan Mathematical Journal* 9.4 (1962).

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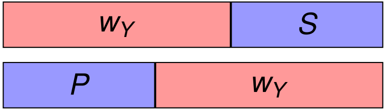
Finally, the word

$$w_Y \triangleq \alpha^M \beta^M \alpha^M \beta^M \alpha^{2M} \beta^{2M} \quad \in L_Y$$

prevents large self-overlaps, where $M = \lceil |M_{\mathrm{Lit}}| / |\alpha| \rceil$ and M_{Lit} is the longest literal in φ.

Lemma

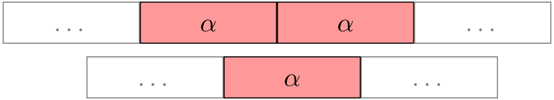
The equation $w_Y S = P w_Y$ has no solutions with $|S| \leq |w_Y| - (M+1)|\alpha|$.



Details on our choice of W_Y

Proof sketch, direction from φ' to φ

To show that w_Y truly has the desired properties we first observe that whenever we consider a long enough overlap, we have α^2 above α (or similarly for β).



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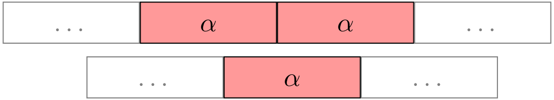

Recall, that since α is primitive, the equation $\alpha^2 = P\alpha S$ has only solutions with $P = \varepsilon$ or $S = \varepsilon$. Therefore, we need to consider overlaps of w_Y with w_Y only with certain granularity.

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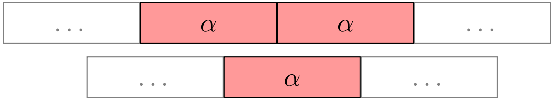

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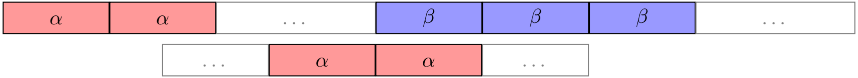

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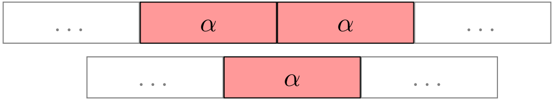

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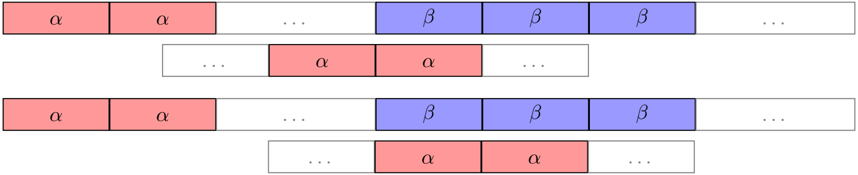

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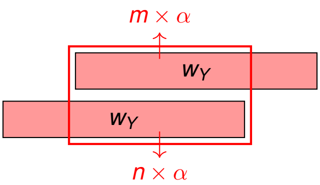
Proof sketch, direction from φ' to φ

For any of such remaining 'granular' overlaps we directly show that whenever we consider an overlap of w_Y with itself, there is a different number of α's in the overlapping portions of w_Y from $\sigma(\mathcal{N})$ and $\sigma(\mathcal{H})$[2].



[2] except in one case

Dealing with situations when Y is on both sides

STEP 2. REMOVING VARIABLES OCCURRING ON BOTH SIDES

Input: A formula

$$\varphi = \neg contains(\mathcal{N}, \mathcal{H}) \wedge \bigwedge_{X \in \mathbb{X}} X \in L_X$$

Output: An equisatisfiable formula

$$\varphi' = \neg contains(\mathcal{N}', \mathcal{H}') \wedge \bigwedge_{X \in \mathbb{X}} X \in L_X$$

such that every non-flat variable Y occurring both in $\mathcal{N}$ and $\mathcal{H}$ has been replaced by a corresponding $\square_Y$, yielding $\mathcal{N}'$ and $\mathcal{H}'$. I.e. we iteratively replace suitable variables by fresh symbols.

$$\neg contains(\mathcal{N}, \mathcal{H}) \land \bigwedge_{X \in \mathbb{X}} X \in L_X$$

How to handle non-flat variables occurring only in $\mathcal{H}$?

Non-flat variables occurring only in $\mathcal{H}$

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Lemma

Let $Y \in \mathbb{X}$ be a non-flat variable, and let

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There is a formula φ' equisatisfiable to φ such that

$$\varphi' = \neg contains(\mathcal{N}, v_0 Y v_1 \cdots Y v_n) \wedge Y \in L'_Y \wedge \bigwedge_{X \in \mathbb{X} \setminus \{Y\}} X \in L_X$$

with $L'_Y \subseteq L_Y$ being a flat language.

Non-flat variables occurring only in $\mathcal{H}$

Naive approach

Again, assume a partial assignment $\sigma' \colon \mathbb{X} \setminus \{Y\} \to \Sigma^*$.

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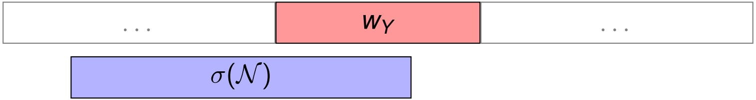
A naive approach would be to enumerate $w \in L_Y$, and check whether $\sigma \triangleq \sigma' \triangleleft \{Y \mapsto w\}$ is a model.



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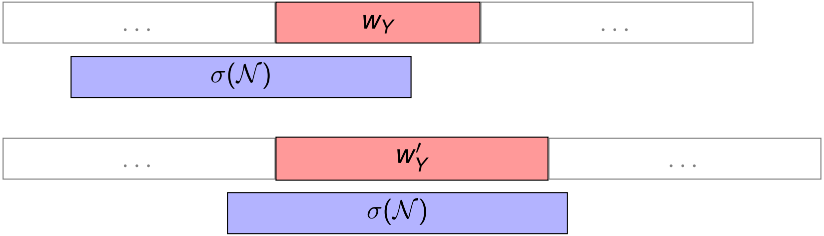

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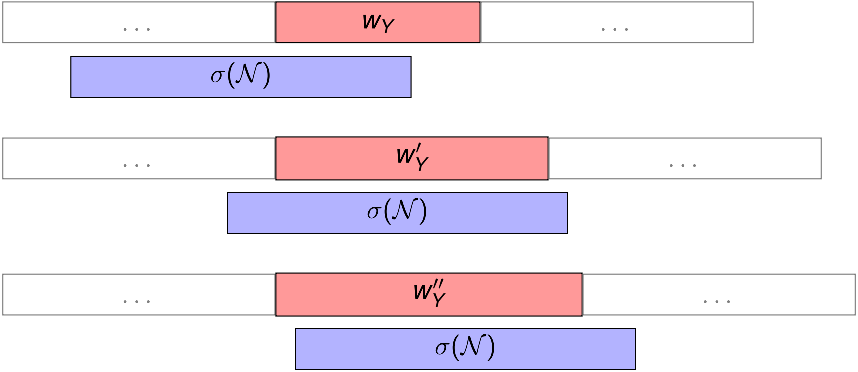

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It would be much easier to solve modified formulae

$$\varphi_{Pref} \triangleq \neg contains(\mathcal{N}[Y/Y_p\#], \mathcal{H}[Y/Y_p\#]),$$
$$\varphi_{Suf} \triangleq \neg contains(\mathcal{N}[Y/\#Y_s], \mathcal{H}[Y/\#Y_s]).$$

where $\#$ is a fresh separator symbol and Y_p (Y_s) is restricted to prefixes (suffixes) of Y.

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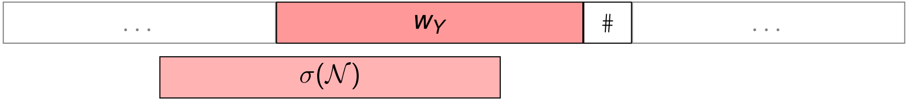
Intuitively, if $\sigma \not\models \varphi_{Pref}$ then we have the following situation:



Modularizing the proof

We introduce Γ_Y—a tool that allows us to solve φ_{Pref} and φ_{Suf} separately[3] and then glue together the prefix and suffix to produce $\sigma \models \neg contains(\mathcal{N}, \mathcal{H})$.

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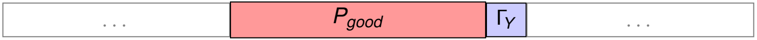

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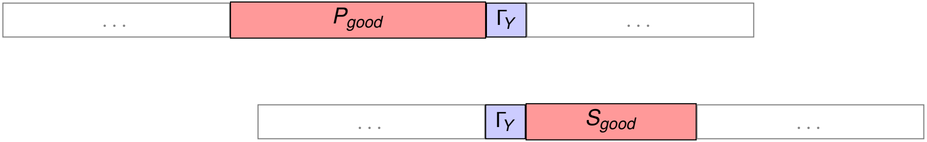

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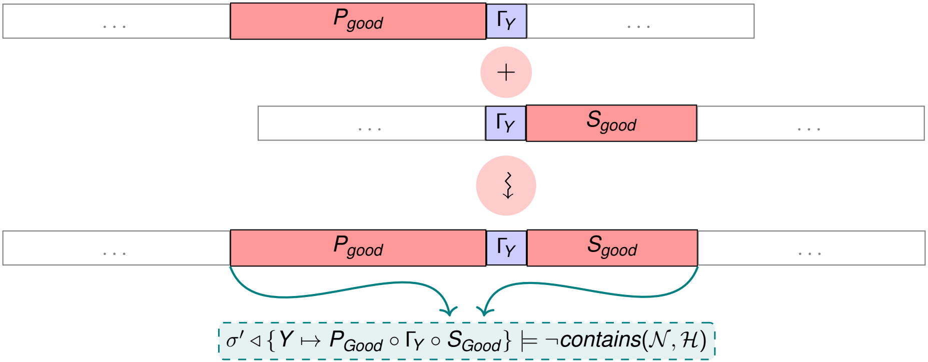
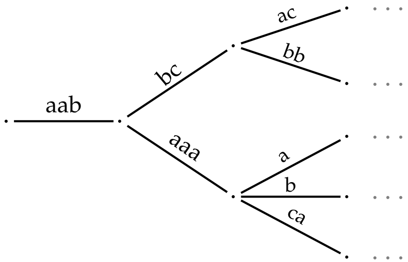


[3] With some technical assumptions on σ'

Finding a suitable prefix

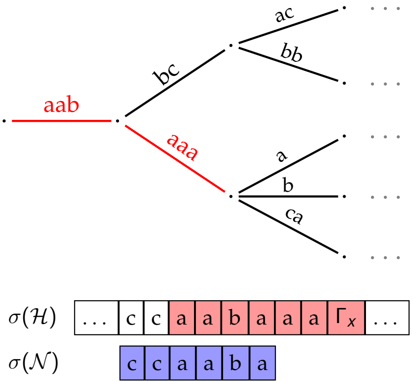
We explore prefixes of Y systematically, using a *prefix tree*.



Finding a suitable prefix

Some vertices are dead ends

Consider the prefix aabaaa, and the following situation.



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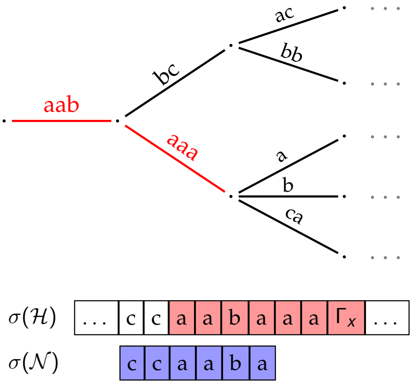
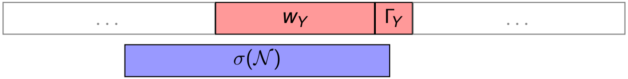


We mark some nodes as dead ends, and do not explore their successors.

Failure to find a good prefix

Special form of a solution

We explore prefixes in the prefix tree up to a certain bound λ.
It is useful to think of Γ_Y as a new alphabet symbol, however, it is still just a word.



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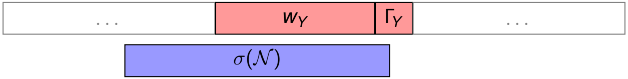


Therefore, after exploring the prefix tree up to λ, we might be in a situation:

- We have not found a good prefix, and
- there are vertices (leading to unexplored prefixes longer than λ) that are not dead-ends.

Failure to find a good prefix

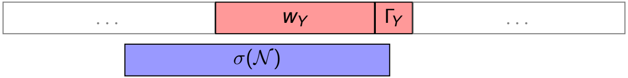
Special form of a solution

[4] Technical assumption. Justification: variables with a sorter value can be replaced by their values.

Failure to find a good prefix

Special form of a solution

Let us analyse a prefix w_Y with $|w_Y| > \lambda$ that leads to a vertex that is not marked as a dead end.



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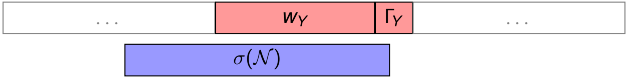

Moreover, we have the following:

1. $\mathcal{N}$ contains only flat variables, and
2. $\sigma(X)$ is longer than some constant for every flat variable X[4].

---

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Special form of a solution (continued)

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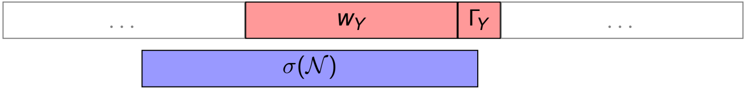

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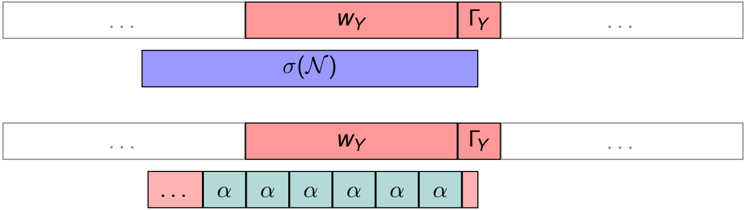

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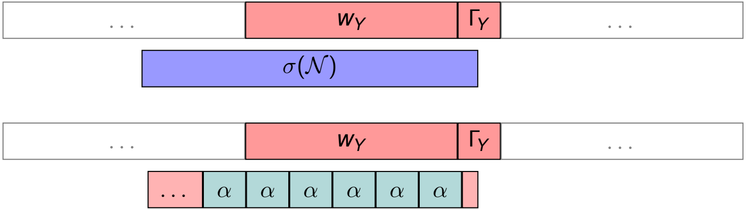

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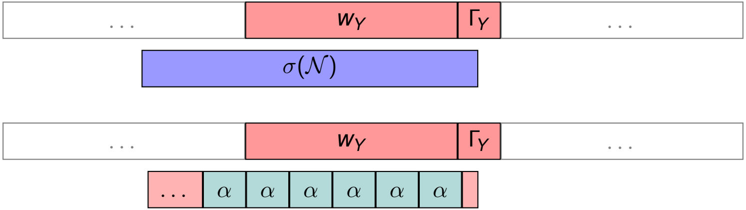

Therefore, $W_Y = s \circ \alpha^k \circ p$ for some $p \in Pref(\alpha)$, $s \in Suf(\alpha)$ and $k \in \mathbb{N}$ where $L_X = (\alpha^\ell)^*$ is the language of the rightmost variable in $\mathcal{N}$.

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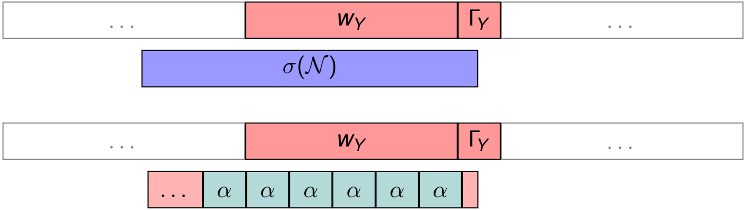

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We show that if there is a model $\sigma \triangleq \sigma' \triangleleft \{Y \mapsto w_Y\}$ with $|w_Y| > \lambda$ such that w_Y has the form $w_Y = p\alpha^k s$, then there is a model $\hat{\sigma} = \sigma' \triangleleft \{Y \mapsto w'_y\}$ with $w'_Y \in L'_Y$.

$$L'_Y \triangleq (p\alpha^* s\Gamma_Y) \cap L_Y$$

Producing a complete flat underapproximation

A complete flat underapproximation of a non-flat language L_Y is computed as

$$L'_Y \triangleq F_Y \cup \left(\mathrm{Glue}(P_Y, S_Y) \cap L_Y\right)$$

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4. $\mathrm{Glue}(p \circ \Gamma_Y, \Gamma_Y \circ s) \triangleq p \circ \Gamma_Y \circ s$

Applying underapproximation

STEP 3. UNDERAPPROXIMATE NON-FLAT VARIABLES OCCURRING ONLY $\mathcal{N}$

Input: A formula

$$\varphi \triangleq \neg contains(\mathcal{N}, \mathcal{H}) \land \bigwedge_{X \in \mathbb{X}} X \in L_X$$

with $\mathcal{N}$ containing only flat variables and a set $\mathbb{X}_{\mathcal{N}}$ of non-flat variables occurring in $\mathcal{N}$.

Output: An equisatisfiable formula

$$\varphi \triangleq \neg contains(\mathcal{N}, \mathcal{H}) \land \bigwedge_{X \in \mathbb{X} \setminus \mathbb{X}_{\mathcal{N}}} X \in L_X \land \bigwedge_{Y \in \mathbb{X}_{\mathcal{N}}} Y \in L'_Y$$

such that the language L'_Y is flat for every $Y \in \mathbb{X}_{\mathcal{N}}$.

Entire decision procedure (sketch)

1. Normalize φ_0 into a disjunction $\bigvee_{i \in I} \varphi_i$, pick a disjunct $\varphi_i = \neg \mathit{contains}(\mathcal{N}_i, \mathcal{H}_i) \land \ldots$.

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Future work:

- Extend our proof to cover conjunctions of $\neg$*contains*.
- Improve the complexity bounds of our result.

Conclusion: Although $\neg$*contains* hides quantifiers inside, it is *decidable*.

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Thank you for you attention.
Questions?