Intermission — Numbers

or: what really happened in the Jedi High Council room

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Jedi High Council room.

Enter Yoda and Anakin.

Anakin: Master Yoda, I've finished my training and I'm ready to start the trials to become a Jedi.

Master Yoda: Impatient be not, young Padawan. Not you is it who decides whether ready you are.

Anakin: But Master Yoda, I've passed all the required tests and built my own lightsaber; I can control the Force.

Master Yoda: Speak no more, young one. Control the Force can you? Even me not dare say so. What knowledge acquired have you?

Anakin: I've learnt the history of the Jedi order, all languages of the Republic, and know about all inhabited systems in the galaxy.

Master Yoda: Tell me, apprentice, about numbers what do you know?

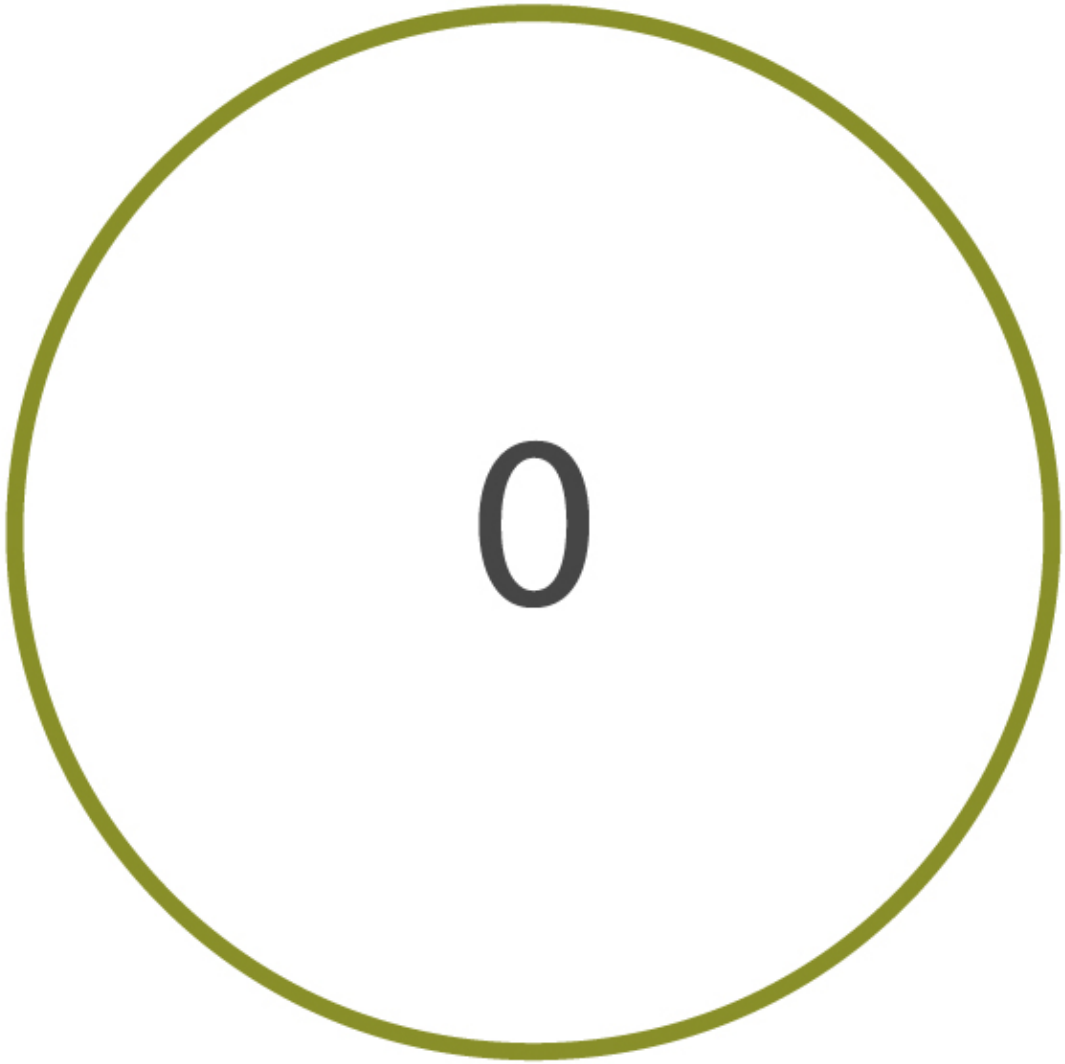
Anakin: Numbers? There are many of them, you know, zero, one, two, three, . . .

Master Yoda: Excuse me, young Padawan, but recognize those “zero, one, . . .” things I do not. When given a thing, to tell whether a number it is can I how?

Anakin: Okay, Master Yoda, just assume there are these things, “*numbers*”, and zero is one them.

Master Yoda: Of course! Zero a number is. That means that a number exists, right I am?

Numbers



Anakin: Yes, but not only one. Master Yoda, every number has a successor, and that successor also have a successor, and so on. Now, one is the successor of zero, $S(0)$, two is the successor of one, $S(S(0))$, etc., indefinitely, because we can take the successor of any number.

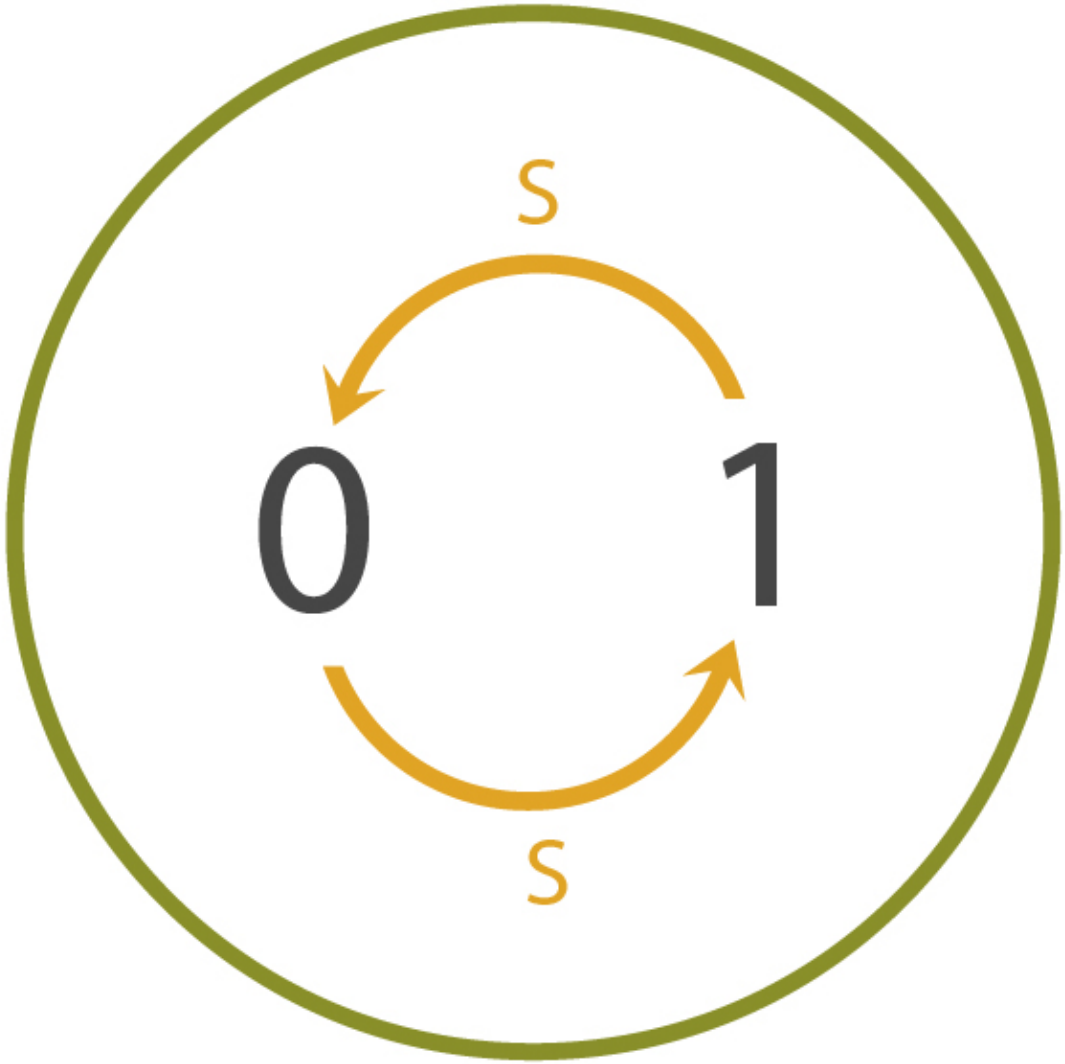
Master Yoda: In other words, the successor of any number also a number is.

Anakin: Exactly!

Master Yoda: Ahaa. In a simple case, thinking I am, two equal to zero we would have.

Anakin: What? No, why would that be...

Numbers



Master Yoda: Visualizing a case when two numbers successors of each other are I was. One number successor of itself I visualized could have; trivial too much such a case would be, though.

Anakin: Wait, Master Yoda, that's not the model of the numbers!

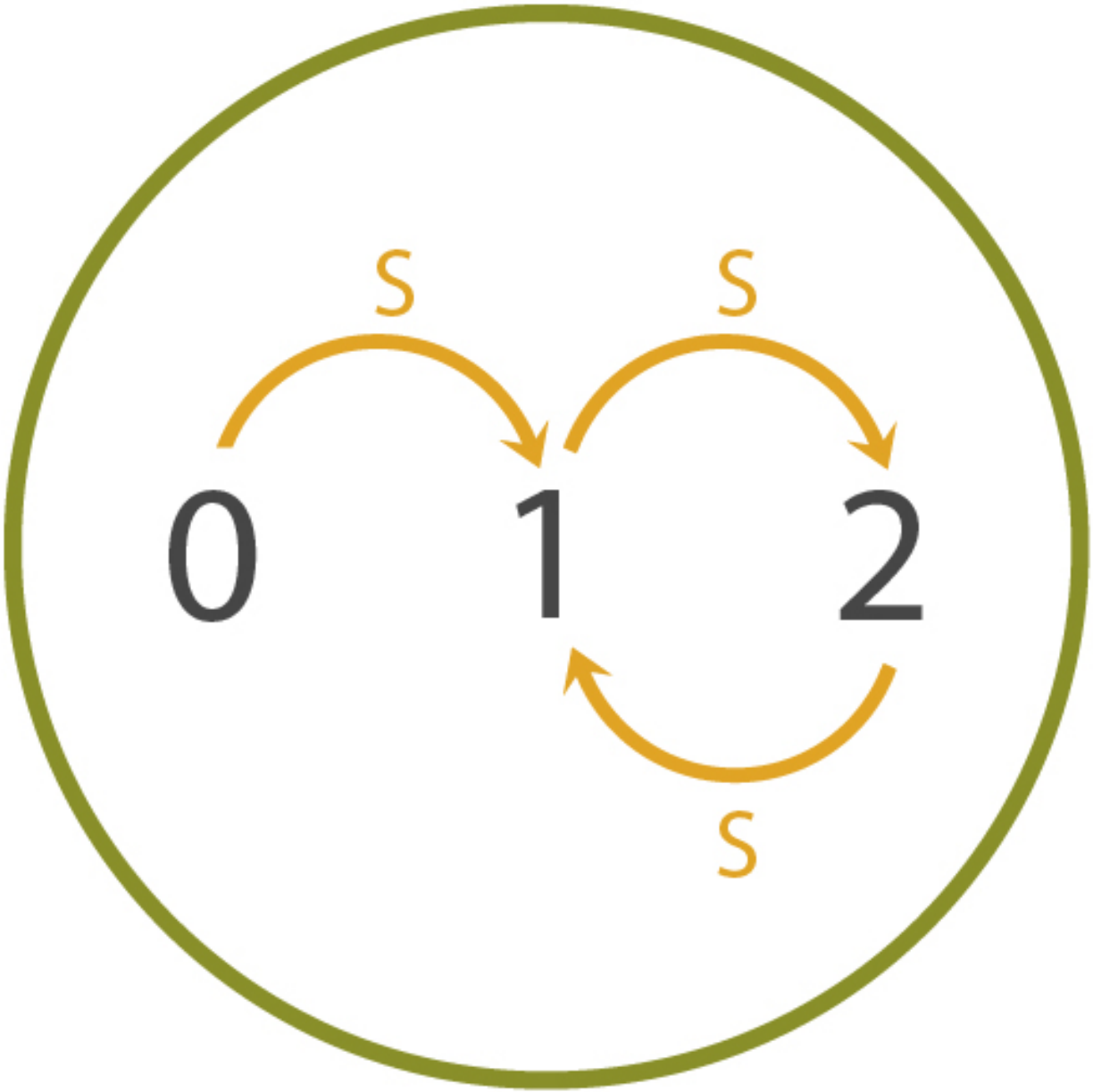
Master Yoda: Why not? What property do numbers have that model this does not?

Anakin: Because, um. . . zero is not the successor of any number. Your model has a successor link from 1 to 0, and that's not allowed.

Master Yoda: So $S(S(0)) = 0$ we cannot have. Still, $S(S(S(0))) = S0$ possible is.

Anakin: What? How. . .

Numbers

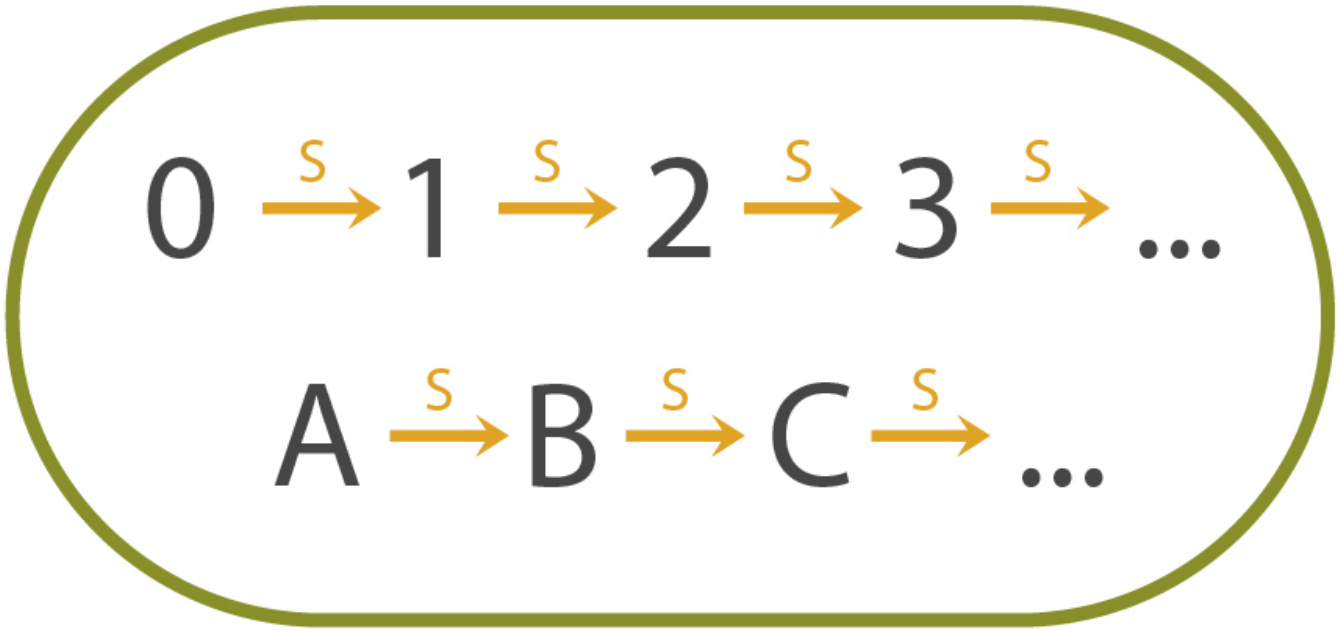


Anakin: No, no! Because. . . *[looking into a textbook]* if two numbers have the same successor, they are the same number, that's why! 2 and 0 both cannot have 1 as a successor, unless they're the same number. If 2 was the same number as 0, then 1's successor would be 0, and that's not allowed, because 0 is no the successor of any number!

Master Yoda: Open my eyes have you now, young Padawan. A multitude of numbers exist, then. The first chain forever on goes.

Anakin: It seems you are starting to get what I — wait. Hold on. What do you mean by "*the first chain*". . .

Numbers

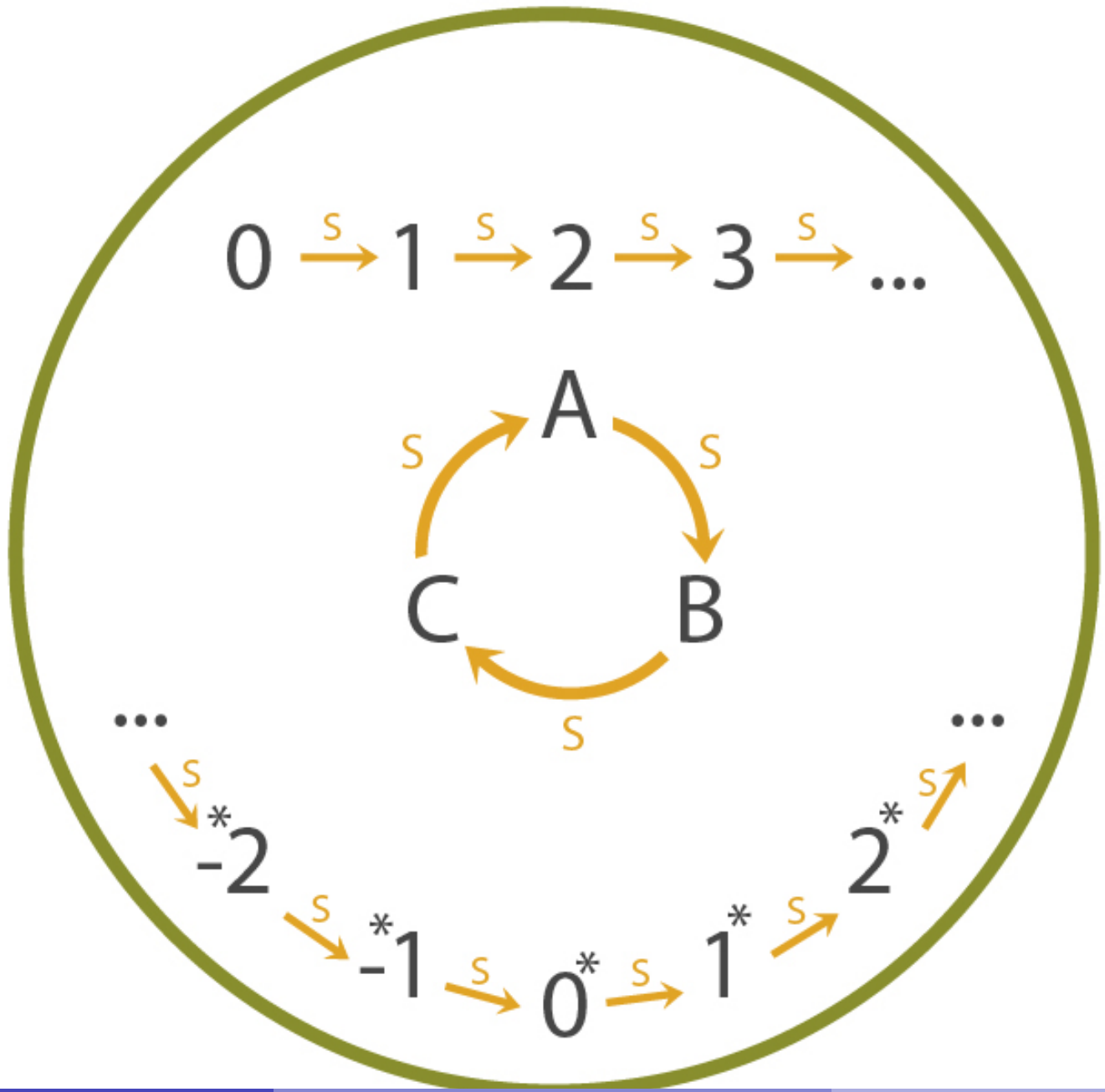


Master Yoda: At least one start of an infinite chain, called zero, was there, said you.

Anakin: I misspoke. Zero is the *only* number that is not a successor of any number.

Master Yoda: Understand now I do. Then any other chain either looping is, or forever in *both* directions on goes.

Anakin: Wha?


Numbers
0 S 1 S 2 S 3 S ...
A
S
S
C
B
S
...
S
*
-2
S
*
-1
S
0*
S
1*
S
2*
S
...

Master Yoda: (i) Zero the only number not a successor of other number is, (ii) successor of a number a number is, (iii) if numbers two the same successor have, the same number they are. Following rules those, successor chains all besides the one from zero starting loop or forever go in both directions need to.

Anakin: But, Master Yoda, there aren't supposed to be any chains besides the one that starts at zero! Especially the looping chain looks just weird!

Master Yoda: Looping chains you like not, young Anakin? Another rules then you need.

Anakin: But how can I get rid of them in general? I can remove the 3-loop by saying that $\nexists x : S(S(S(x))) = x$, but such a rule removes only one case and looks quite arbitrary. I would need infinitely many rules like these.

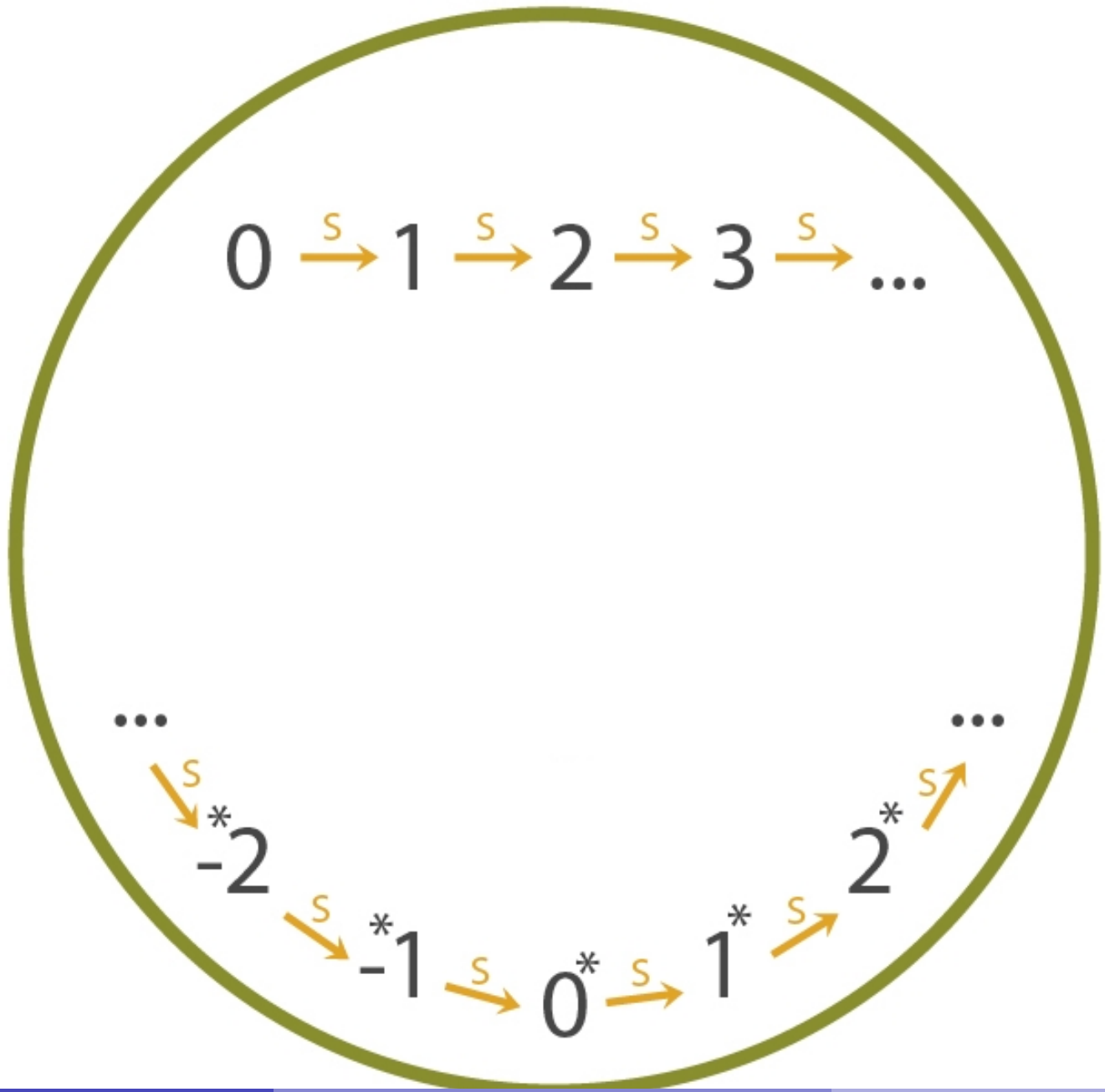
Master Yoda: The following rule create you can. For $\varphi(x)$ a formula with free variable single, let φ of 0 true be, i.e., $\varphi(0)$. Also, let φ for a successor of any number where it holds hold, $\forall x : \varphi(x) \to \varphi(S(x))$. Then, φ for all numbers true is, $\forall n : \varphi(n)$.

Anakin: But how can this rule help me get rid of the loops?

Master Yoda: If the formula $\varphi \equiv x \neq S(S(S(x)))$ you take, looping chains of three remove then you can. Similarly, N-loops for all N you remove.

Anakin: I see! So now, the only elements left are in the one-direction infinite chain starting from zero, right?

Master Yoda: Not so fast...


Numbers
0 S→ 1 S→ 2 S→ 3 S→ ...
... S→ -2* S→ -1* S→ 0* S→ 1* S→ 2* S→ ...

Anakin: But what do we need to get rid of the bi-directional infinite chain of nonstandard numbers?

Master Yoda: In first-order logic, those chains remove you cannot. Second-order logic, the logic of properties, would we need:

$$\forall_2 P : P(0) \wedge (\forall_1 x : P(x) \rightarrow P(S(x))) \rightarrow (\forall_1 n : P(n))$$

Anakin: Then, let's also use this rule!

Master Yoda: Be careful, young Anakin. Trudging into the area of second-order logic a dangerous choice is...

Anakin: I don't see you concerns Master Yoda. Second-order logic is *so* powerful! I'll use it to save Padmé! Don't worry, I'll be careful!

Master Yoda: ...

References

[Eliezer Yudkowsky. Logical Pinpointing. URL: http://lesswrong.com/lw/f4e/logical_pinpointing/]

[Eliezer Yudkowsky. Standard and Nonstandard Numbers. URL: http://lesswrong.com/lw/g0i/standard_and_nonstandard_numbers/]