Automata-based decision procedures

IAM, Lecture 4

Lukáš Holík

Reminder: Presburger arithmetic

Is interpreted over $\mathbb{N}$, has the signature

$$\{0, S, +, =\}$$

Part I

Formulae as automata

Numbers as words

- in last significant bit first encoding (LSBF)

0	is encoded as	0
1	is encoded as	1
2	is encoded as	01
10	is encoded as	0101

...

Numbers as words

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$$\begin{array}{rl} 0 & \text{is encoded as } 0 \\ 1 & \text{is encoded as } 1 \\ 2 & \text{is encoded as } 01 \\ 10 & \text{is encoded as } 0101 \\ & \dots \end{array}$$

- also, every word from $w0^*$ denotes the same number as w

010
0100
01000 all encode 2.
01000000000
...

Assignments as words

- assignments seen as k-tuples of numbers
  (+ an ordering on its k free variables)

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- $L(\varphi)$ denotes all encodings of all satisfying assignments of φ

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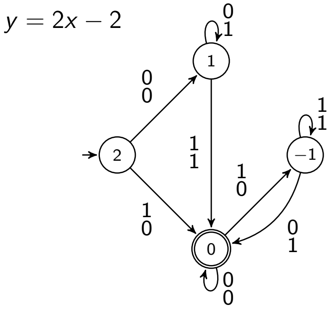

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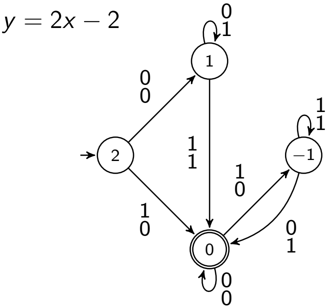

- To decide satisfiability of a formula φ
  - construct an automaton A with $L(A) = L(\varphi)$
  - and test emptiness of its language.

Deciding Presburger using automata

- Build the FA with $L(A) = L(\varphi)$ inductively to φ's structure.

$$\neg\,(x \geq y) \;\wedge\; \exists z.\,(z \leq x+4 \vee \exists w.\, x < w < y)$$

Deciding Presburger using automata

- Build the FA with $L(A) = L(\varphi)$ inductively to φ's structure.

$$\neg \underbrace{(x \geq y)}_{A_3} \wedge \exists z.\, (\underbrace{z \leq x+4}_{A_2} \vee \exists w.\, \underbrace{x < w < y}_{A_1})$$

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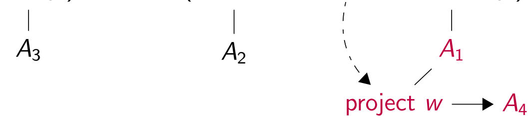

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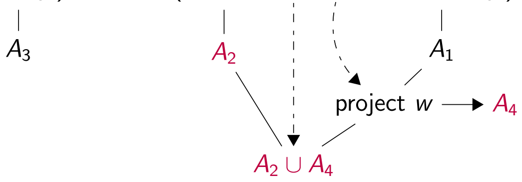

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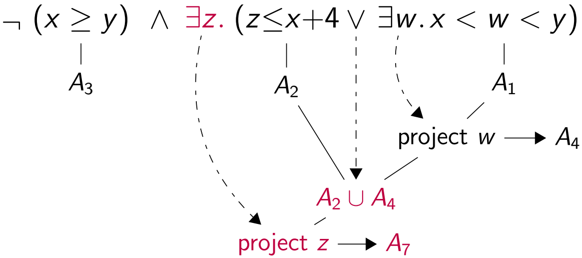

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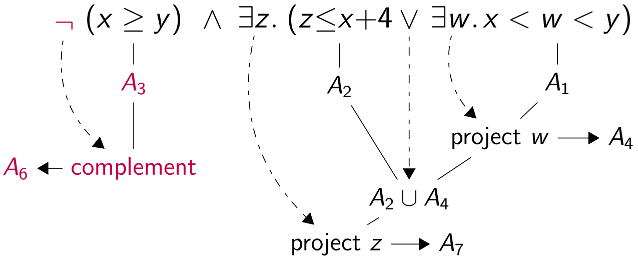

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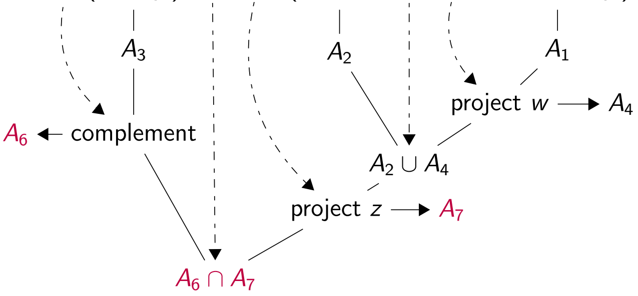

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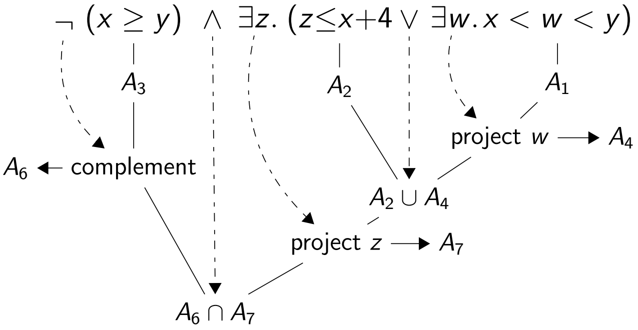


- Then check language emptiness of $A = A_6 \cap A_7$.

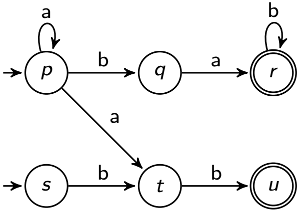
Ingredients

1. automata for atomic predicates
2. automata constructions for $\cup, \cap, \neg, \exists$
3. automata language emptiness test

Part II

Automata crash course

Finite automaton $A = (\Sigma, Q, I, \delta, F)$



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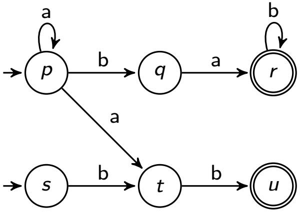

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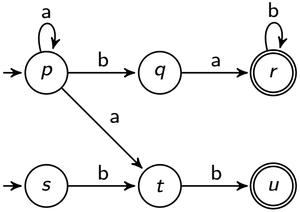

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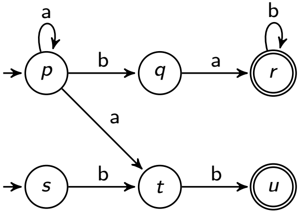

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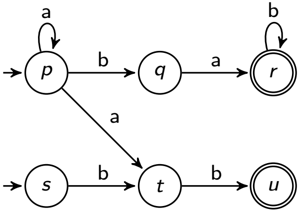

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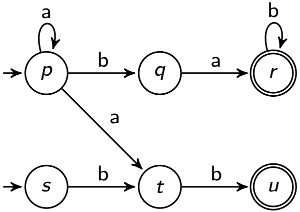

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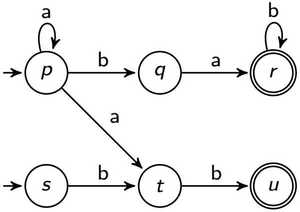

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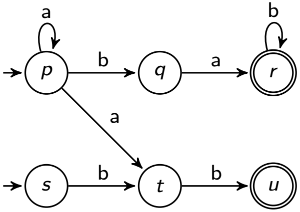

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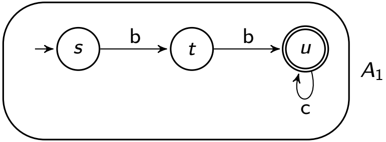
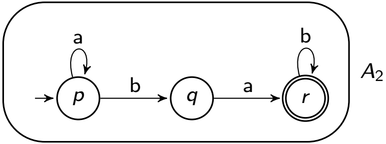
- a word is read in a run, which can be accepting or rejecting
- accepts a word if it has some accepting run over it
- language $L(A)$ is the set of all accepted words
- can be deterministic or nondeterministic
  - deterministic has at most one run for every word

Automata union, $\cup$

- We need $L(A_1) \cup L(A_2) = L(A_1 \cup A_2)$
- Simply unite the automata.





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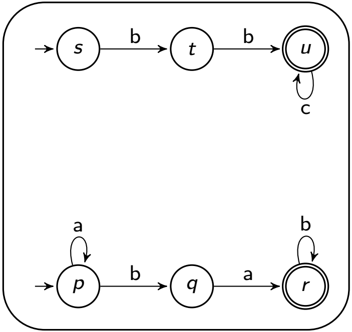

$A_1 \cup A_2$

Assume $Q_1 \cap Q_2 = \emptyset$.
$Q' = Q_1 \cup Q_2$
$I' = I_1 \cup I_2$
$F' = F_1 \cup F_2$
$\delta' = \delta_1 \cup \delta_2$

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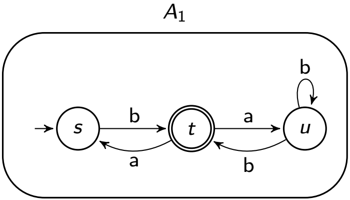

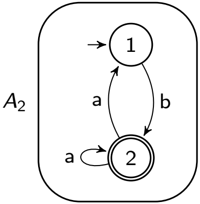

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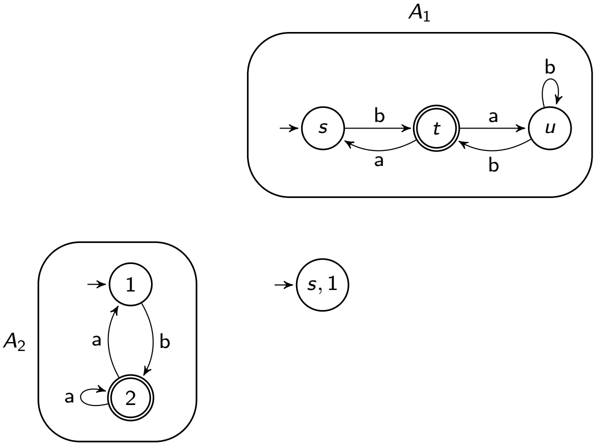

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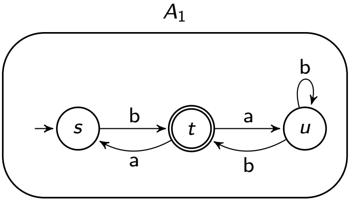

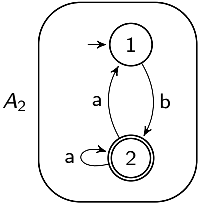

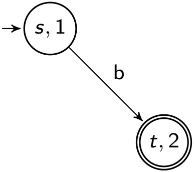

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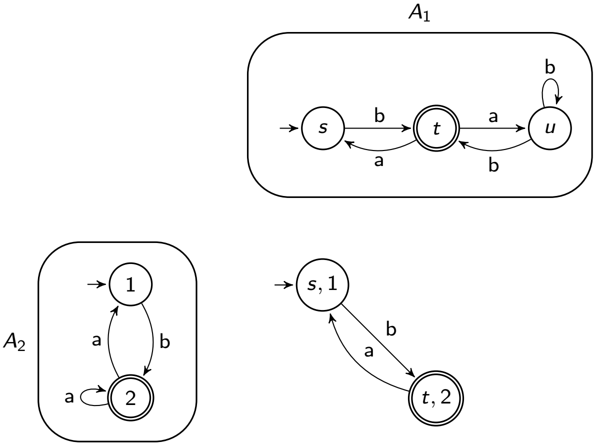

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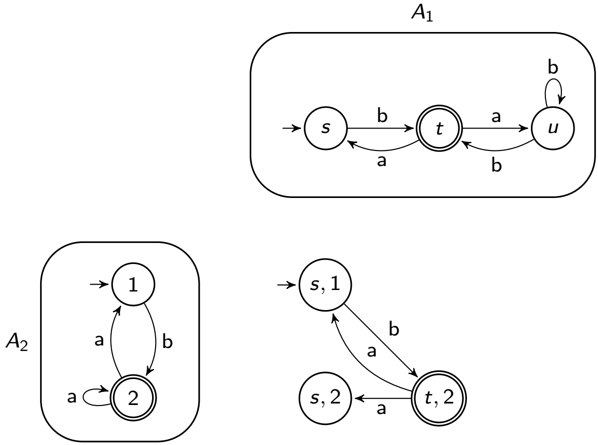

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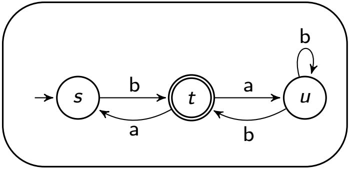
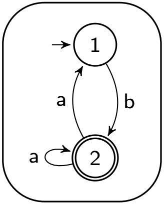
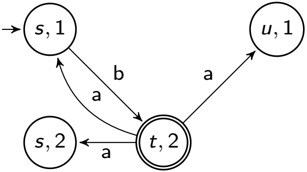


A_2





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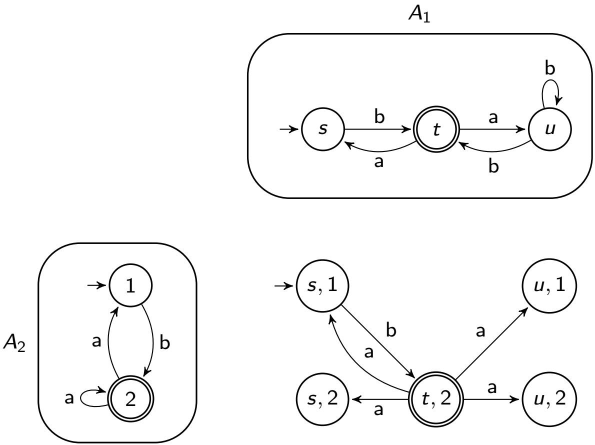

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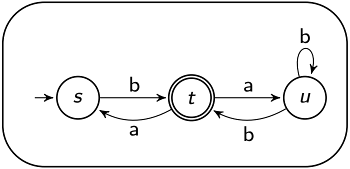

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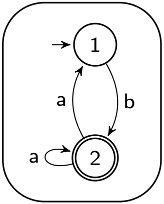

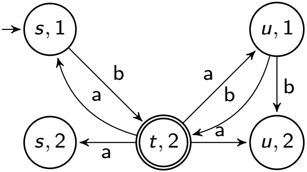

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\begin{aligned}
Q' &= Q_1 \times Q_2 \\
I' &= I_1 \times I_2 \\
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\delta'((q, r), a) &= \delta(q, a) \times \delta(r, a)
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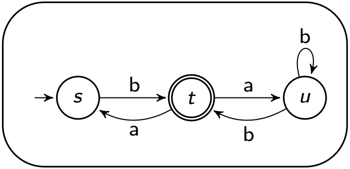
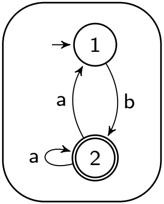
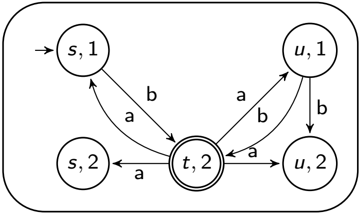


A_2





$A_1 \cap A_2$

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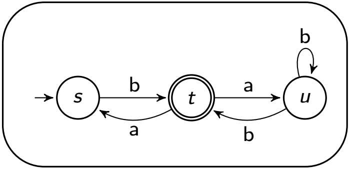

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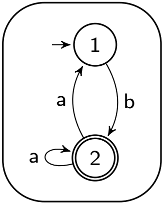

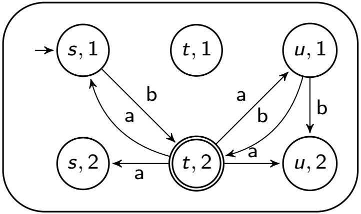

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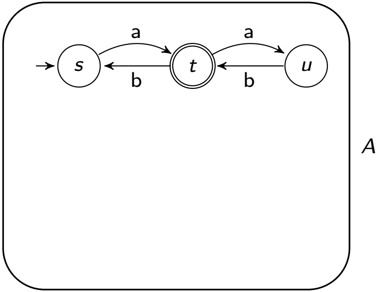
Automata complement, $\neg$

- We need $\Sigma^* \setminus L(A) = L(\neg A)$
- If deterministic, complete and negate acceptance.



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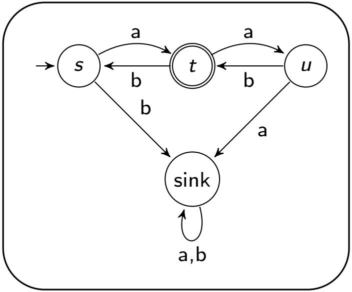

A completed

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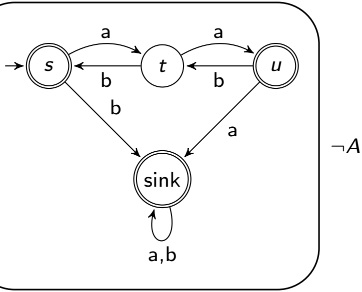
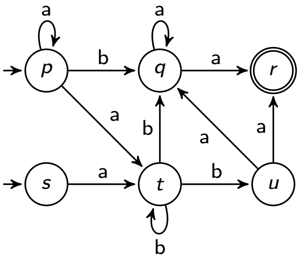
Assume A determ. complete.
$Q' = Q$
$I' = I$
$F' = Q \setminus F$
$\delta' = \delta$



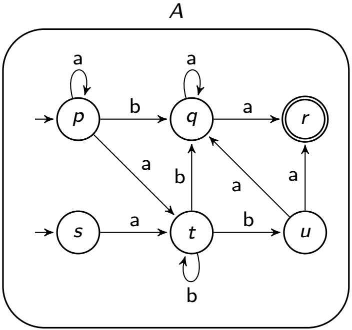
Complement has a problem with nondeterminism



- accepting as well as rejecting runs over *aba*
- hence *aba* is in $L(A)$ and stays after negating acceptance
- determinisation is needed

Determinization

- We need a deterministic A' with $L(A') = L(A)$.
- Subset construction.



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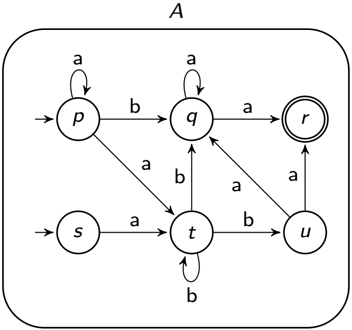

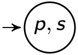

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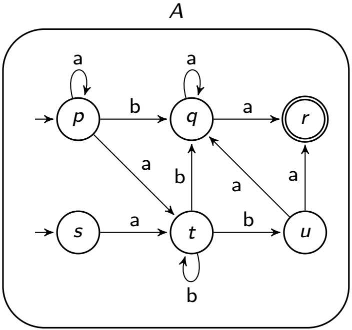

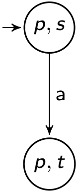

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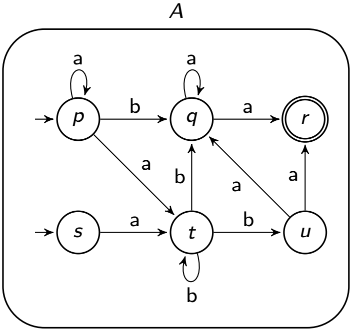

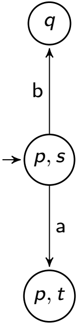

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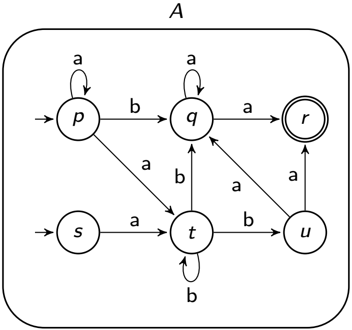

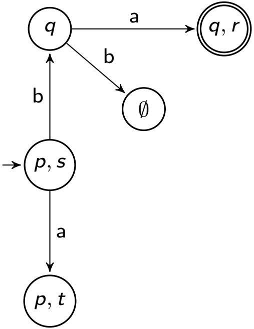

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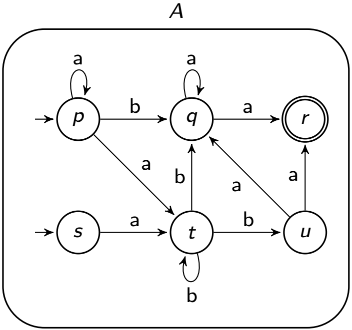

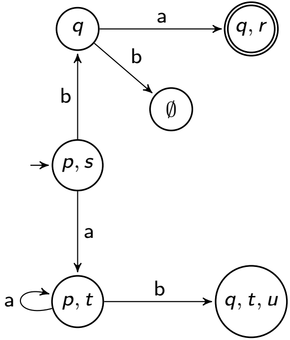

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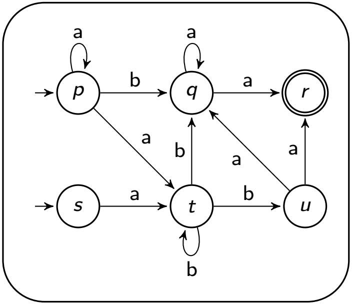
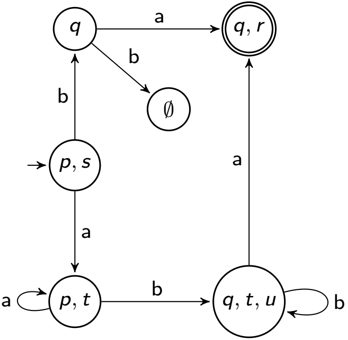
A





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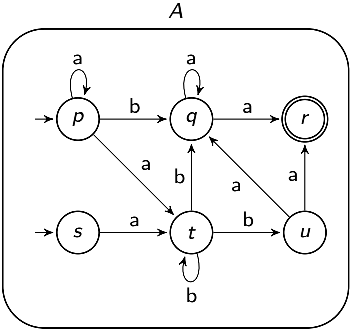

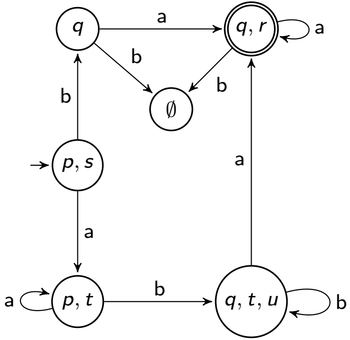

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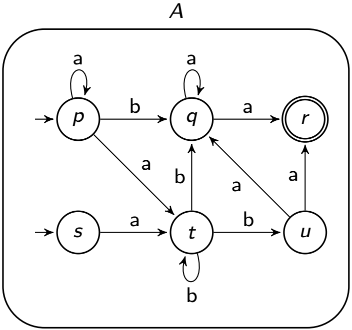

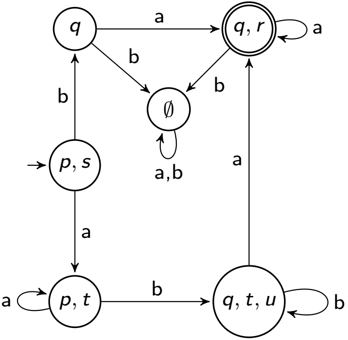

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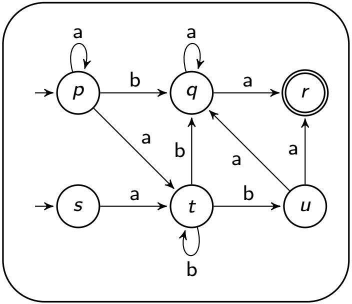
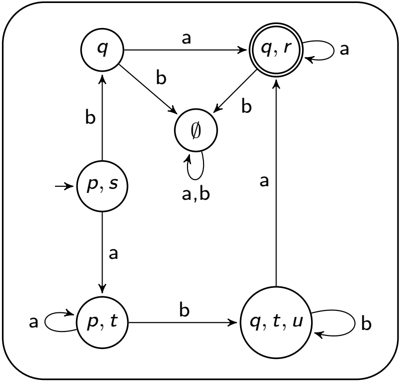


$Q' = 2^Q$ $\quad F' = \{S \in Q' \mid S \cap F \neq \emptyset\}$

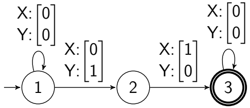
$I' = \{I\}$ $\quad \delta'(S, a) = \bigcup_{s \in S} \delta(s, a)$

A' = determinized A



Automata projection, $\exists x$

- Remove the x track (project on the y track).



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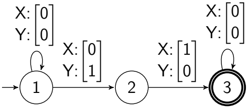
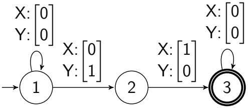


A



$\rightarrow$ Projection on X

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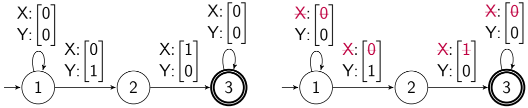

A $\qquad$ $\rightarrow$ Projection on X

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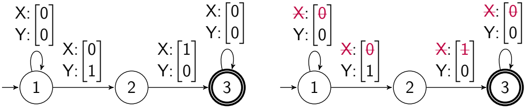
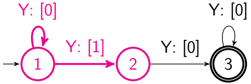


A $\quad\rightarrow$ Projection on X

- Careful, does it accept all encodings of sat. assignments?



$10, 010, 0010, \dots$ are accepted,
$1, 01, 001, \dots$ should be accepted too

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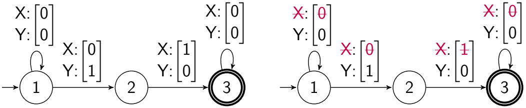

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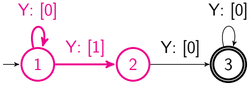

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- Saturation acceptance: everything reaching final state by zero vectors becomes also accepting.

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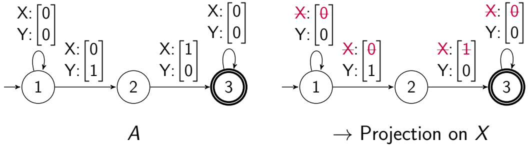

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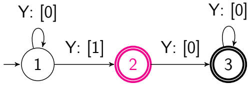

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Part III

Automata for Atomic Presburger Predicates

Atomic predicates

- Assume that atomic predicates were transformed into the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, \ldots, a_n, b \in \mathbb{Z}$.

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$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

  where $a_1, \ldots, a_n, b \in \mathbb{Z}$.
- We write

$$\bar{a} \cdot \bar{x} = b$$

  where $\bar{a} = (a_1, \ldots, a_n)$, $\bar{x} = (x_1, \ldots, x_n)$, and $\bar{a} \cdot \bar{x}$ denotes the scalar product.

States of atomic predicates

Example: $2x - y = 2$

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Idea:

- The states of the automaton are numbers $q \in \mathbb{Z}$.
- From q will be read those assignment to x and y under which $2x - y$ equals q.

States of atomic predicates

Example: $2x - y = 2$

Idea:

- The states of the automaton are numbers $q \in \mathbb{Z}$.
- From q will be read those assignment to x and y under which $2x - y$ equals q.

From state $q \in \mathbb{Z}$ are read encodings of $\bar{c} \in \mathbb{N}^n$ such that $\bar{a} \cdot \bar{c} = q$.

Initial states of atomic predicates $2x - y = 2$

Reminder:
from q read $\bar{c}$ s.t. $\bar{a} \cdot \bar{c} = q$

- The whole assignment is read from the initial state.

Initial states of atomic predicates $\quad 2x - y = 2$

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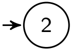

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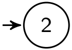

2 is the only initial state

Transitions of atomic predicates $\quad 2x - y = 2$

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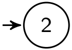
- If $q \xrightarrow{\zeta} q'$ then:



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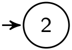

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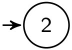

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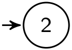

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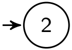

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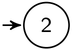

$$\delta(q,\zeta) = \begin{cases} \{\frac{1}{2}(q - \bar{a} \cdot \zeta)\} & \text{if } q - \bar{a} \cdot \zeta \text{ is even} \\ \emptyset & \textit{otherwise} \end{cases}$$

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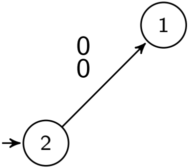

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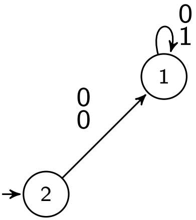

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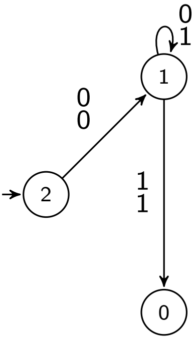

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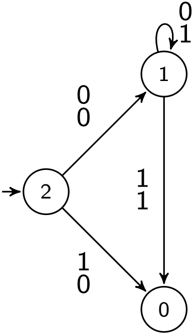

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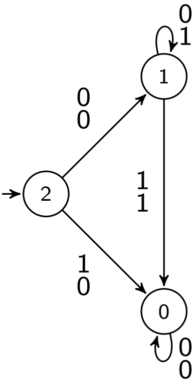

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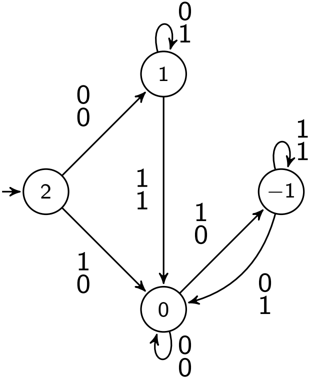
- $\bar{c}'$ is read from q'
- iff $2\bar{c}' + \zeta$ read from q
- iff $\bar{a} \cdot (2\bar{c}' + \zeta) = q$
- iff $\bar{a} \cdot \bar{c}' = \frac{1}{2}(q - \bar{a} \cdot \zeta)$

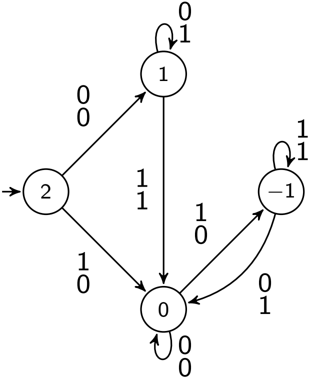


$$\delta(q,\zeta) = \begin{cases} \{\frac{1}{2}(q - \bar{a} \cdot \zeta)\} & \text{if } q - \bar{a} \cdot \zeta \text{ is even} \\ \emptyset & \textit{otherwise} \end{cases}$$

Accepting states of atomic predicates $2x - y = 2$

Reminder:
from q read $\bar{c}$ s.t. $\bar{a} \cdot \bar{c} = q$

- Accepting are those where nothing (i.e. ϵ) needs to be read to satisfy $\bar{a} \cdot \bar{c} = q$.



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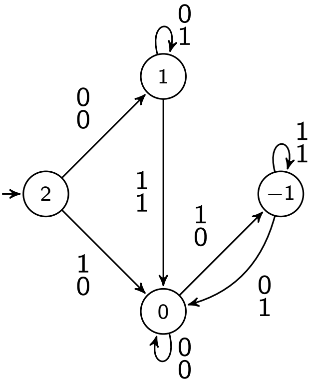

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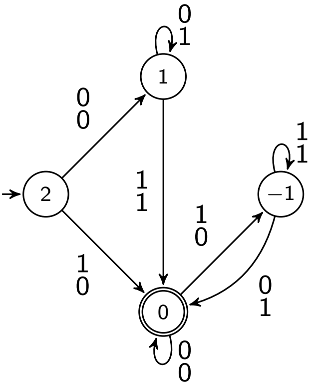

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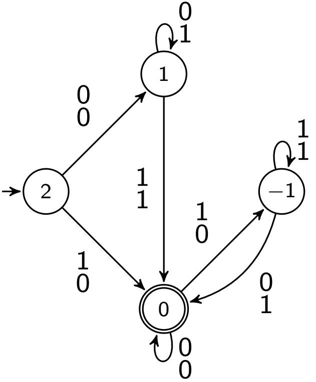

0 is the only accepting state

Atomic predicates: algorithm

EqtoDFA(φ)
Input: Equation $\varphi = \; a \cdot x = b$
Output: DFA $A = (Q, \Sigma, \delta, q_0, F)$ such that $L(A) = L(\varphi)$ (without trap state)

$Q, \delta, F \leftarrow \emptyset$; $q_0 \leftarrow s_b$
$W \leftarrow \{s_b\}$
while $W \neq \emptyset$ **do**
    pick s_k **from** W
    add s_k **to** Q
    if $k = 0$ **then add** s_k **to** F
    for all $\zeta \in \{0, 1\}^n$ **do**
        if $(k - a \cdot \zeta)$ is even **then**
            $j \leftarrow \frac{1}{2}(k - a \cdot \zeta)$
            if $s_j \notin Q$ **then add** s_j **to** W
            add (s_k, ζ, s_j) **to** δ

Remarks

- We have seen a procedure quite different from the ones based on quantifier elimination.
- It can be optimized, extended to $\mathbb{Z}$.
- It shows how diverse solutions of a problem can be,
- and a surprising connection between arithmetic and automata.
- Integer/Presburger arithmetic are somewhat “regular”.

Part IV

Weak Monadic Second Order Logic of One Successor (WS1S)

WS1S

- Minimalistic syntax

$$\varphi \to X \subseteq X \mid succ(X) \mid \varphi \vee \varphi \mid \neg\varphi \mid \exists X.\varphi$$

- interpreted over finite subsets of $\mathbb{N}$.

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- $Partition(X, X_1, \ldots, X_n)$:
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- Also Presburger arithmetic!

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- **second order** = can quantify over sets
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- **monadic** = only sets numbers, not relations
- **one successor** = the *succ* function

Why

- Looks different, but is surprisingly close to Automata, Presburger, regularity.
- It is The basic automata logic. Starting point for many other interesting automata-related logics.
- Exciting research!

Assignments as words

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- An assignment, n-tuple of sets, becomes a word over $\{0,1\}^n$.
  $X = \{1,3,4\}$, $Y = \{2\}$, $Z = \emptyset$

$$\begin{matrix}01011\\00100\\00000\end{matrix}, \begin{matrix}01011\,0\\00100\,0\\00000\,0\end{matrix}, \begin{matrix}01011\,00\\00100\,00\\00000\,00\end{matrix}, \begin{matrix}01011\,000\\00100\,000\\00000\,000\end{matrix}, \cdots$$

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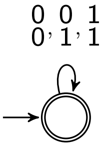

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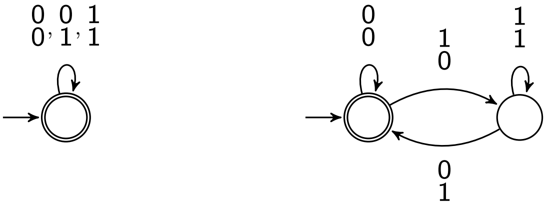

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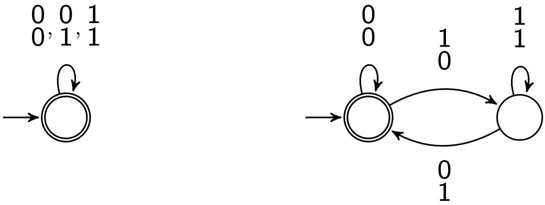

- and everything else is the same as for Presburger!

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- Regularity for words: WS1S $=$ automata [Büchi 1960] $=$ regular expressions $=$ Presburger with bit-subset
- Complexity
  - Presburger with automata (a bit different algo.): $\mathcal{O}(2^{2^{2^n}})$
  - WS1S: non-elementary complexity $\mathcal{O}(\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}}_{\text{alternations}})$