Lecture 3 — First-Order Theories

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First-Order Theories

First-Order Theories

- When reasoning in first-order logic (FOL), we use theories to add semantics to function/predicate symbols.
- A theory restricts the possible interpretations of a formula to those we are interested in.

Example

Is the following

$$\varphi : 1 + 1 = 2$$

a valid FOL formula? Why?

- Validity: φ is valid iff $I \models \varphi$ for all interpretations I.
- There are interpretations for which the formula is not true
  - e.g., $I = \{\mathbb{N}, \alpha_I\}$ s.t. $\alpha_I(+) = \{\ldots, (1, 1) \mapsto 3, \ldots\}$
- We wish to restrict possible interpretations of $\varphi \rightsquigarrow$ **theories**.

Theories

Theory $\mathcal{T}$ is defined using

- **signature** $\Sigma_\mathcal{T}$: set of function and predicate symbols
  - note that *constants* are special function symbols!
  - $\Sigma_\mathcal{T}$-formula: a formula over $\Sigma_\mathcal{T}$
- **axioms** $\mathcal{A}_\mathcal{T}$: set of *closed* FOL formulae over the vocabulary of $\Sigma_\mathcal{T}$
  - often, we need an *infinite* number of axioms $\rightsquigarrow$ axiom schemata
  - axiom schema — a *template* whose instantiations produce axioms
  - can be seen as a *program* that generates axioms or determines whether a formula is an axiom
  - axioms are used to **restrict** possible interpretations of formulae to *interesting* ones
- We use **FOL($\mathcal{T}$)** to denote FOL over $\Sigma_\mathcal{T}$ with axioms from $\mathcal{A}_\mathcal{T}$.

Fragment of a theory:

- a syntactically restricted subset of formulae of the theory
- e.g., the quantifier-free fragment, alternation-free fragment, fragments restricting the number of quantifier alternations, ...
- we often show equivalence of fragments of FOL with other formal models

$\mathcal{T}$-validity and $\mathcal{T}$-satisfiability

$\mathcal{T}$-validity and $\mathcal{T}$-satisfiability:

- $\mathcal{T}$-interpretation: an interpretation I that satisfies all axioms of $\mathcal{T}$:

$$I \models A \qquad \text{for every } A \in \mathcal{A}_{\mathcal{T}}.$$

- A $\Sigma_{\mathcal{T}}$-formula φ is $\mathcal{T}$-valid if it holds for every $\mathcal{T}$-interpretation.
  - we denote $\mathcal{T}$-validity as $\mathcal{T} \models \varphi$
- A $\Sigma_{\mathcal{T}}$-formula φ is $\mathcal{T}$-satisfiable if there is a $\mathcal{T}$-interpretation for which it is true.

Completeness and Consistency

Completeness:

- A theory $\mathcal{T}$ is complete if for every *closed* $\Sigma_{\mathcal{T}}$-formula φ,

$$\text{either} \quad \mathcal{T} \vdash \varphi \quad \text{or} \quad \mathcal{T} \vdash \neg\varphi$$

  ($\mathcal{T} \vdash \varphi$ means "φ is provable in $\mathcal{T}$").
  - Can be seen as whether the axiomatization restricts interpretations in the right way.

- Do not confuse with the completeness of *proof systems*!
  - (A proof system S for FOL is *complete* if for every FOL formula φ such that $\models \varphi$, it holds that $\vdash_S \varphi$.)

Consistency:

- A theory $\mathcal{T}$ is consistent if there is at least one $\mathcal{T}$-interpretation.
- Alternative definition: A theory is inconsistent if for every $\Sigma_{\mathcal{T}}$-formula φ it holds that $\mathcal{T} \vdash \varphi$, otherwise it is consistent.

Decidability

Decidability

- a **theory** $\mathcal{T}$ is decidable if there is an algorithm that for every $\Sigma_{\mathcal{T}}$-formula φ terminates with "**yes**" if $\mathcal{T} \models \varphi$ and with "**no**" if $\mathcal{T} \not\models \varphi$ (and the algorithm always terminates).
- FOL($\emptyset$), i.e. FOL without any theory, is **undecidable**
- a **fragment** of $\mathcal{T}$ is decidable if it is decidable for each formula φ that obeys the fragment's syntactic restrictions.
- quantifier-free fragment:
  - validity/satisfiability in FOL are defined for ground formulae only
  - satisfiability: when testing satisfiability, a quantifier-free formula is prefixed by existential quantification of free variables
  - validity: when testing validity, a quantifier-free formula is prefixed by universal quantification of free variables

Theory of Equality $\mathcal{T}_E$

Theory of Equality $\mathcal{T}_E$ (with Uninterpreted Functions):

- Signature: $\{=, f, g, h, \ldots, p, q, r, \ldots\}$
  - equality $(=)/2$ and all function and predicate symbols
- Axioms:
  1. $\forall x \,.\, x = x$ (reflexivity)
  2. $\forall x, y \,.\, x = y \quad \rightarrow \quad y = x$ (symmetry)
  3. $\forall x, y, z \,.\, x = y \wedge y = z \quad \rightarrow \quad x = z$ (transitivity)
  4. for every positive integer n and n-ary function symbol f,

     $$\forall \overline{x}, \overline{y} \,.\, \left(\bigwedge_{i=1}^{n} x_i = y_i \right) \quad \rightarrow \quad f(\overline{x}) = f(\overline{y}) \qquad \text{(function congruence)}$$

  5. for every positive integer n and n-ary predicate symbol p,

     $$\forall \overline{x}, \overline{y} \,.\, \left(\bigwedge_{i=1}^{n} x_i = y_i \right) \quad \rightarrow \quad (p(\overline{x}) \leftrightarrow p(\overline{y})) \qquad \text{(predicate congruence)}$$

  $\overline{x}$ denotes a list of variables $x_1, \ldots, x_n$
- Note that only the $(=)$ predicate symbol is interpreted.
- Note that [4] and [5] are *axiom schemata*.

Theory of Equality $\mathcal{T}_E$

- **undecidable**: it allows all functions and predicates
  - (any FOL formula can be encoded into $\mathcal{T}_E$)
- the quantifier-free fragment is **decidable**
  - using the congruence closure algorithm
- $\mathcal{T}_E$ is often used as a part of other theories
  - some definitions of FOL treat $(=)$ as a *special* predicate

Example

The formula

$$f(f(f(a))) = a \quad \wedge \quad f(f(f(f(f(a))))) = a \quad \wedge \quad f(a) \neq a$$

is unsatisfiable.

Peano Arithmetic $\mathcal{T}_{PA}$

Peano Arithmetic $\mathcal{T}_{PA}$ (first-order arithmetic):

- Signature: $\{0, S, +, \cdot, =\}$
  - $0/0$ is a constant (nullary functions)
  - $S/1$ is a unary function symbol (called *successor*)
  - $(+)/2$ and $(\cdot)/2$ are binary function symbols
  - equality $(=)/2$ is a binary predicate symbol
- Axioms:
  1. $\forall x.\ \neg(S(x) = 0)$ (zero)
  2. $\forall x, y.\ S(x) = S(y) \quad \rightarrow \quad x = y$ (successor)
  3. for every $\Sigma_{\mathcal{T}_{PA}}$-formula φ with precisely one free variable,

$$\big(\varphi(0) \wedge (\forall x.\ \varphi(x) \rightarrow \varphi(S(x)))\big) \quad \rightarrow \quad \forall x.\ \varphi(x) \qquad \text{(induction)}$$

  4. $\forall x.\ x + 0 = x$ (plus zero)
  5. $\forall x, y.\ x + S(y) = S(x + y)$ (plus successor)
  6. $\forall x.\ x \cdot 0 = 0$ (times zero)
  7. $\forall x, y.\ x \cdot S(y) = x \cdot y + x$ (times successor)

Peano Arithmetic $\mathcal{T}_{\mathsf{PA}}$

- Intended interpretations:
  - standard meaning of the function and predicate symbols over $\mathbb{N}$

Example ($\leq$)

We can define inequality $\leq$ using the following equivalence:

$$x \leq y \qquad \Leftrightarrow \qquad \exists z \,.\, x + z = y.$$

Example

$$\exists x, y, z \,.\, x \neq 0 \wedge y \neq 0 \wedge z \neq 0 \wedge xx + yy = zz$$

Peano Arithmetic $\mathcal{T}_{\mathrm{PA}}$

undecidable

Theorem (Gödel's First Incompleteness Theorem (Gödel 1931))

Every consistent recursive FOL theory that contains $\mathcal{T}_{\mathrm{PA}}$ is incomplete.

Notes:

- recursive theory: there is an algorithm that will, given a formula φ, decide whether φ is an axiom of the theory
  - all commonly considered theories are recursive
- therefore, if $\mathcal{T}_{\mathrm{PA}}$ is consistent, there is a $\Sigma_{\mathcal{T}_{\mathrm{PA}}}$-formula φ such that

$$\text{neither} \quad \mathcal{T}_{\mathrm{PA}} \vdash \varphi \quad \text{nor} \quad \mathcal{T}_{\mathrm{PA}} \vdash \neg\varphi$$

- therefore, every sufficiently strong formal system (in particular, a system with arithmetic) is either inconsistent or incomplete

Peano Arithmetic $\mathcal{T}_{\mathsf{PA}}$

Proof. (high-level idea).

- Words over an alphabet Σ can be encoded as numbers in $\mathcal{T}_{\mathsf{PA}}$.

Example

Let $\Sigma = \{a, b, c\}$ and let $\# : \Sigma \to \mathbb{N}$ be injective, e.g., $\#(a) = 2, \#(b) = 3, \#(c) = 4$. Then the number

$$2^{\#(a)} \cdot 3^{\#(b)} \cdot 5^{\#(c)} \cdot 7^{\#(b)} \cdot 11^{\#(a)} = 2,801,452,500$$

uniquely encodes the string "$abcba$".

- Therefore, any formula φ can also be encoded as a number.
  - called its Gödel number $\mathcal{G}(\varphi)$
- A proof $P \rightsquigarrow$ also a number $\mathcal{G}(P)$.
- Application of proof rules $\rightsquigarrow$ manipulation with numbers.
- Consider the formula $\alpha(x, y)$ that encodes the statement

$$\alpha(x, y) \quad \overset{\text{def}}{\Leftrightarrow} \quad \mathcal{G}^{-1}(x) \text{ is a proof of the formula } \mathcal{G}^{-1}(y).$$

Peano Arithmetic $\mathcal{T}_{\mathsf{PA}}$

Proof. (cont.)

$$\alpha(x,y) \quad \overset{\text{def}}{\Leftrightarrow} \quad \mathcal{G}^{-1}(x) \text{ is a proof of the formula } \mathcal{G}^{-1}(y).$$

- Now, take the formula

$$Bew(y) \quad \overset{\text{def}}{\Leftrightarrow} \quad \exists x\,.\,\alpha(x,y)$$

expressing "$\mathcal{G}^{-1}(y)$ is a provable *(beweisbar)* formula" ($\vdash \mathcal{G}^{-1}(y)$)

- Note that $\mathcal{G}^{-1}(y)$ is provable iff $Bew(y)$ is provable.
- Consider the following statement:

Gödel's Statement

$$\varphi \quad \overset{\text{def}}{\Leftrightarrow} \quad \neg Bew(\mathcal{G}(\varphi))$$

$$\models \varphi \quad \Rightarrow \quad \nvdash \varphi$$
$$\models \neg\varphi \quad \Rightarrow \quad \vdash \varphi$$

"φ is true iff φ is unprovable."

- Generalization of the "Liar's paradox." (diagonalization) □

Peano Arithmetic $\mathcal{T}_{\mathsf{PA}}$

Gödel's Incompleteness$_2$ and Completeness$_1$ Theorems:

Theorem (Gödel's Completeness$_1$ Theorem)

FOL with the semantic argument proof system is complete$_1$.

- The theorem also holds for any other *standard* proof system:
  - Hilbert system, natural deduction, ...
- Completeness: two different meanings, complete$_1$ and complete$_2$
  - G's Completeness$_1$ T.: a system S is complete$_1$ if for any φ s.t. $\models \varphi$ it holds that $\vdash_S \varphi$.
  - G's Incompleteness$_2$ T.: a theory $\mathcal{T}$ is complete$_2$ if for any *closed* $\Sigma_{\mathcal{T}}$-formula φ, either $\mathcal{T} \vdash \varphi$ or $\mathcal{T} \vdash \neg\varphi$.
- G's Incompleteness$_2$ T. says the following:

  G's Statement (GS) is neither provable nor disprovable in PA.
- Therefore, by G's Completeness$_1$ T., there are models of PA where GS is false. But GS is true in the standard model.
- $\leadsto$ there exist **nondstandard models of Peano Arithmetic**

Presburger Arithmetic $\mathcal{T}_{\mathbb{N}}$

Presburger Arithmetic $\mathcal{T}_{\mathbb{N}}$:

- Signature: $\{0, S, +, =\}$
  - $0/0$ and is a constant (nullary functions)
  - $S/1$ is a unary function symbol (called *successor*)
  - $(+)/2$ and is a binary function symbol
  - equality $(=)/2$ is a binary predicate symbol
- Axioms (a subset of Peano arithmetic):
  1. $\forall x.\ \neg(S(x) = 0)$ (zero)
  2. $\forall x, y.\ S(x) = S(y) \quad \rightarrow \quad x = y$ (successor)
  3. $\big(\varphi(0) \wedge (\forall x.\ \varphi(x) \rightarrow \varphi(S(x)))\big) \quad \rightarrow \quad \forall x.\ \varphi(x)$ (induction)
  4. $\forall x.\ x + 0 = x$ (plus zero)
  5. $\forall x, y.\ x + S(y) = S(x + y)$ (plus successor)

Presburger Arithmetic $\mathcal{T}_{\mathbb{N}}$

- intended interpretations:
  - standard meaning of the function and predicate symbols over $\mathbb{N}$
- **decidable** [Presburger 1929]
- decision procedures:
  - quantifier elimination-based
  - automata-based
- it is easy to extend to integers $\mathbb{Z}$

Example

The following formula over $\mathbb{Z}$

$$\forall x, z \, \exists y \, . \, 2x - y = 3z + 5$$

can be written when using variables over $\mathbb{N}$ as

$$\forall x_p, x_n, z_p, z_n \, \exists y_p, y_n \, . \, 2(x_p - x_n) - (y_p - y_n) = 3(z_p - z_n) + 5.$$

Expressed in $\mathcal{T}_{\mathbb{N}}$ by moving negative terms to the other side:

$$\forall x_p, x_n, z_p, z_n \, \exists y_p, y_n \, . \, 2x_p + y_n + 3z_n = 3z_p + 5 + 2x_n + y_p.$$

Theory of Integers $\mathcal{T}_{\mathbb{Z}}$

Theory of Integers $\mathcal{T}_{\mathbb{Z}}$:

- Signature:
  $\{\ldots, -2, -1, 0, 1, 2, \ldots, (-3\cdot), (-2\cdot), (2\cdot), (3\cdot), \ldots, +, -, =, <\}$
  - $\ldots, -2, -1, 0, 1, 2, \ldots$ are constants intended to be assigned to the obvious values in $\mathbb{Z}$
  - $\ldots, (-2\cdot), (-1\cdot), (1\cdot), (2\cdot), \ldots$ are unary functions intended to be assigned to constant coefficients
  - $(+)/2$ and $(-)/2$ are binary function symbols intended to represent $+_{\mathbb{Z}}$ and $-_{\mathbb{Z}}$ respectively
  - $(=)/2$ and $(<)/2$ are binary predicate symbols intended to represent $=_{\mathbb{Z}}$ and $<_{\mathbb{Z}}$ respectively
- Every $\Sigma_{\mathcal{T}_{\mathbb{Z}}}$-formula can be reduced to $\Sigma_{\mathcal{T}_{\mathbb{N}}}$.

Theory of Reals $\mathcal{T}_{\mathbb{R}}$

Theory of Reals $\mathcal{T}_{\mathbb{R}}$ (elementary algebra):

- Signature: $\{0, 1, +, \cdot, -, =, \leq\}$
  - $0/0$ and $1/0$ are constants
  - $(+)/2$ and $(\cdot)/2$ are binary function symbols
  - $(-)/1$ is a unary function symbol (additive inverse)
  - $(=)/2$ and $\leq /2$ are binary predicate symbols

- Axioms: the axioms are split into several groups

- Axioms of an **abelian group**:
  1. $\forall x, y, z.\ (x + y) + z = x + (y + z)$ ($+$ associativity)
  2. $\forall x.\ x + 0 = x$ ($+$ identity)
  3. $\forall x.\ x + (-x) = 0$ ($+$ inverse)
  4. $\forall x, y.\ x + y = y + x$ ($+$ commutativity)

Theory of Reals $\mathcal{T}_{\mathbb{R}}$

- Additional axioms of a **ring**:
  1. $\forall x, y, z.\ (xy)z = x(yz)$ ($\cdot$ associativity)
  2. $\forall x.\ x1 = x$ ($\cdot$ right identity)
  3. $\forall x.\ 1x = x$ ($\cdot$ left identity)
  4. $\forall x, y, z.\ x(y + z) = xy + xz$ ($\cdot$ left distributivity over $+$)
  5. $\forall x, y, z.\ (x + y)z = xz + yz$ ($\cdot$ right distributivity over $+$)

- Additional axioms of a **field**:
  1. $\forall x, y.\ xy = yx$ ($\cdot$ commutativity)
  2. $0 \neq 1$ (separate identities)
  3. $\forall x.\ x \neq 0 \quad \rightarrow \quad \exists y.\ xy = 1$ ($\cdot$ inverse)

Theory of Reals $\mathcal{T}_{\mathbb{R}}$

- Axioms of a **total order**:
  1. $\forall x, y.\ x \leq y \wedge y \leq x \quad \rightarrow \quad x = y$ (antisymmetry)
  2. $\forall x, y, z.\ x \leq y \wedge y \leq z \quad \rightarrow \quad x \leq z$ (transitivity)
  3. $\forall x, y.\ x \leq y \vee y \leq x$ (totality)

- Additional axioms of a **real closed field**:
  1. $\forall x, y, z.\ x \leq y \quad \rightarrow \quad x + z \leq y + z$ ($+$ ordered)
  2. $\forall x, y.\ 0 \leq x \wedge 0 \leq y \quad \rightarrow \quad 0 \leq xy$ ($\cdot$ ordered)
  3. $\forall x\ \exists y.\ x = y^2 \vee x = -y^2$ (square root)
  4. for every odd integer n,

$$\forall \overline{x}\ \exists y.\ y^n + x_1 y^{n-1} + \cdots + x_{n-1} y + x_n = 0 \qquad \text{(at least one root)}$$

Theory of Reals $T_{\mathbb{R}}$

- **decidable** [Tarski 1956]
  - via quantifier elimination

Example

Can you find a quantifier-free formula $T_{\mathbb{R}}$-equivalent to the formula

$$\exists x \, . \, ax^2 + bx + c = 0?$$

Solution: the formula

$$b^2 - 4ac \geq 0.$$

Theory of Rationals $T_{\mathbb{Q}}$

Theory of Rationals $T_{\mathbb{Q}}$:

- Signature: $\{0, 1, +, -, =, \leq\}$
  - (same as for $T_{\mathbb{R}}$ excluding $(\cdot)/2$)
- Axioms:
  1. $\forall x, y.\ x \leq y \land y \leq x \quad \rightarrow \quad x = y$ (antisymmetry)
  2. $\forall x, y, z.\ x \leq y \land y \leq z \quad \rightarrow \quad x \leq z$ (transitivity)
  3. $\forall x, y.\ x \leq y \lor y \leq x$ (totality)
  4. $\forall x, y, z.\ (x + y) + z = x + (y + z)$ ($+$ associativity)
  5. $\forall x.\ x + 0 = x$ ($+$ identity)
  6. $\forall x.\ x + (-x) = 0$ ($+$ inverse)
  7. $\forall x, y.\ x + y = y + x$ ($+$ commutativity)
  8. $\forall x, y, z.\ x \leq y \quad \rightarrow \quad x + z \leq y + z$ ($+$ ordered)
  9. for each positive integer n,

     $$\forall x\ .\ nx = 0 \quad \rightarrow \quad x = 0$$ (torsion-free)

  10. for each positive integer n,

      $$\forall x\ \exists y.\ x = ny$$ (divisible)

  where nx denotes $\overbrace{x + \cdots + x}^{n}$

Theory of Rationals $\mathcal{T}_{\mathbb{Q}}$

- **decidable**
  - via quantifier elimination

Example

The formula

$$\frac{1}{2}x + \frac{2}{3}y \geq 4$$

can be expressed as the $\Sigma_{\mathcal{T}_{\mathbb{Z}}}$-formula

$$3x + 4y \geq 24.$$

Example

The formula

$$\exists x \, . \, xx = 2$$

is a valid formula of $\mathcal{T}_{\mathbb{R}}$ but is expressible in neither $\mathcal{T}_{\mathbb{Q}}$ nor $\mathcal{T}_{\mathbb{Z}}$.

Example

The formula

$$\forall x, y \, . \, x < y \quad \rightarrow \quad \exists z \, . \, x < z \wedge z < y$$

is a valid formula of $\mathcal{T}_{\mathbb{R}}$ and $\mathcal{T}_{\mathbb{Q}}$, but an invalid formula of $\mathcal{T}_{\mathbb{N}}$ and $\mathcal{T}_{\mathbb{Z}}$.

Theory of Lists $\mathcal{T}_{\text{List}}$

Theory of Lists $\mathcal{T}_{\text{List}}$:

- Signature: $\{\text{cons}, \text{car}, \text{cdr}, \text{atom}, =\}$
  - $\text{cons}/2$ is a function called the constructor
  - $\text{car}/1$ and $\text{cdr}/1$ are functions called left and right projector
  - $\text{atom}/1$ and $(=)/2$ are predicates
- Axioms:
  1. (reflexivity), (symmetry), and (transitivity) of $\mathcal{T}_E$
  2. instantiations of the (function congruence) axiom scheme of $\mathcal{T}_E$:

$$\forall x_1, x_2, y_1, y_2 \,.\, x_1 = x_2 \wedge y_1 = y_2 \quad \rightarrow \quad \text{cons}(x_1, y_1) = \text{cons}(x_2, y_2)$$
$$\forall x, y \,.\, x = y \quad \rightarrow \quad \text{car}(x) = \text{car}(y)$$
$$\forall x, y \,.\, x = y \quad \rightarrow \quad \text{cdr}(x) = \text{cdr}(y)$$

  3. an instantiation of the (predicate congruence) axiom scheme of $\mathcal{T}_E$:

$$\forall x, y \,.\, x = y \quad \rightarrow \quad (\text{atom}(x) \leftrightarrow \text{atom}(y))$$

  4. $\forall x, y \,.\, \text{car}(\text{cons}(x, y)) = x$ (left projection)
  5. $\forall x, y \,.\, \text{cdr}(\text{cons}(x, y)) = y$ (right projection)
  6. $\forall x \,.\, \neg\text{atom}(x) \quad \rightarrow \quad \text{cons}(\text{car}(x), \text{cdr}(x)) = x$ (construction)
  7. $\forall x, y \,.\, \neg\text{atom}(\text{cons}(x, y))$ (atom)

Theory of Lists $\mathcal{T}_{\text{List}}$

- **undecidable**
- Theory of Acyclic Lists $\mathcal{T}^{+}_{\text{List}}$:
  - created by adding the following axiom schema:

$$
\begin{aligned}
\forall x \,.\, \mathsf{car}(x) &\neq x \\
\forall x \,.\, \mathsf{cdr}(x) &\neq x \\
\forall x \,.\, \mathsf{car}(\mathsf{car}(x)) &\neq x \\
\forall x \,.\, \mathsf{car}(\mathsf{cdr}(x)) &\neq x \\
&\dots
\end{aligned}
$$

  - decidable
- the quantifier-free fragment is **decidable**
- a more general Theory of Recursive Data Structures available

Theory of Arrays $\mathcal{T}_{\mathsf{A}}$

Theory of Arrays $\mathcal{T}_{\mathsf{A}}$:

- Signature: $\{\cdot[\cdot], \cdot\langle\cdot \triangleleft \cdot\rangle, =\}$
  - $\cdot[\cdot]/2$ is a function called the read
  - $\cdot\langle\cdot \triangleleft \cdot\rangle/3$ is a function called the write
  - $(=)/2$ is a predicate
- Axioms:
  1. (reflexivity), (symmetry), and (transitivity) of $\mathcal{T}_E$
  2. $\forall a, i, j \,.\, i = j \quad \rightarrow \quad a[i] = a[j]$ (array congruence)
  3. $\forall a, v, i, j \,.\, i = j \quad \rightarrow \quad a\langle i \triangleleft v\rangle[j] = v$ (read over write 1)
  4. $\forall a, v, i, j \,.\, i \neq j \quad \rightarrow \quad a\langle i \triangleleft v\rangle[j] = a[j]$ (read over write 2)

Theory of Arrays $\mathcal{T}_\mathsf{A}$

- **undecidable**
  - arbitrary functions can be encoded using multi-dimensional arrays
- extended with the (extensionality) axiom, the quantifier-free fragment is decidable

$$\forall a, b \, . \, (\forall i \, . \, a[i] = b[i]) \quad \leftrightarrow \quad a = b \qquad \text{(extensionality)}$$

Example

The formula

$$a[i] = e \quad \rightarrow \quad \forall j \, . \, a\langle i \triangleleft e \rangle[j] = a[j]$$

is $\mathcal{T}_\mathsf{A}$-valid.

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