

Formal Analysis and Verification

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Abstract Interpretation

Introduction

- Compared to model checking in which the stress is put on a systematic execution of a system being verified (or its model), the emphasis in **static analysis** is on minimization of the amount of execution of the code. It is either not executed at all (the case of looking for bug patterns) or just on some abstract level, typically with an in advance fixed abstraction (data flow analysis, **abstract interpretation**, ...).
- However, the borderline between model checking and static analysis is not sharp (especially when considering **abstract interpretation** and model checking based on predicate abstraction).
- Many static analyses are such that they can be applied to parts of code without the need to describe their environment.
- **Static analysis approaches**: bug pattern analysis, type analysis, data flow analysis, ..., **abstract interpretation**, (and sometimes even model checking).

Static Analyses

- Many **different classes** of programs:
 - control-intensive programs,
 - digital signal processing,
 - programs manipulating dynamic memory,
 - programs with integers, . . .
- To make an analysis **efficient** (**effective**), some form of **abstraction** is often needed.
- An analysis **successful** for one class may (and is very likely to) **fail** for a different one (too much imprecision, inefficiency, divergence).
- Analyses are tailored for specific classes of programs
 - the need to prove **soundness** (**completeness**) of each analysis.

Abstract Interpretation — A Motivating Example

Listing 1: SQL Input Sanitization

```
void sanitize(char* query)
{
    for (size_t i = 0; i < strlen(query); ++i)
    {
        if ('\'' == query[i])
            query[i] = '_';
    }

    if (!strcmp(query, "\"\'' OR 1==1\""))
        ERROR();
}
```

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Listing 2: SQL Input Sanitization

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    if (!strcmp(query, "\" OR 1==1"))
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}
```

- The **model checking** approach — check for all values of query \rightsquigarrow **unfeasible!**
- The **abstract interpretation** approach — find a suitable abstract domain for query.

Abstract Interpretation

- Introduced by Patrick and Radhia Cousot at POPL'77.
- A general **framework** for static analyses.
- Concrete analyses are created by providing specific **components** (abstract domain, abstract transformers, ...) to the framework.
- Abstract interpretation transforms a program into an **abstract program** over an **abstract domain** and analyses this (cf. predicate abstraction).
- When certain properties of the components are met, the analysis is guaranteed to be **sound**.

Ingredients of Abstract Interpretation

- **Abstract domain**
 - program states at program locations are represented using **abstract contexts**.
- **Abstract transformers**
 - for each program operation there is a corresponding transformer that represents the effect of the operation performed on an abstract context.
- **Join operator**
 - combines abstract contexts from several branches into a single one.
- **Widening**
 - performed on a sequence of abstract contexts appearing at a given location to accelerate obtaining a **fixpoint**.
- **Narrowing**
 - may be used to refine the result of **widening**.

Abstract Interpretation — formally

- A **semilattice** is a poset with a **join** (least upper bound, supremum) for every finite subset of its base set.

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- A **semilattice** is a poset with a **join** (least upper bound, supremum) for every finite subset of its base set.
- Abstract interpretation I of a program P with the instruction set Instr is a tuple

$$I = (Q, \circ, \sqsubseteq, \top, \perp, \tau)$$

where

- Q is the **abstract domain** (domain of **abstract contexts**),
 - $\top \in Q$ is the supremum of Q ,
 - $\perp \in Q$ is the infimum of Q ,
 - $\circ : Q \times Q \rightarrow Q$ is the **join operator** for accumulation of abstract contexts, (Q, \circ, \top) is a **complete semilattice**,
 - $(\sqsubseteq) \subseteq Q \times Q$ is an ordering defined as $x \sqsubseteq y \iff x \circ y = y$ in (Q, \circ, \top) ,
 - $\tau : \text{Instr} \times Q \rightarrow Q$ defines the interpretation of **abstract transformers**.
-
- The **soundness** of abstract interpretation is guaranteed using **Galois connections**.

Galois Connections

- **Galois connection** is a quadruple $\pi = (\mathcal{P}, \alpha, \gamma, \mathcal{Q})$ such that:
 - $\mathcal{P} = \langle P, \leq \rangle$ and $\mathcal{Q} = \langle Q, \sqsubseteq \rangle$ are *partially ordered sets* (posets),
 - $\alpha : P \rightarrow Q$ and $\gamma : Q \rightarrow P$ are functions such that $\forall p \in P$ and $\forall q \in Q$:

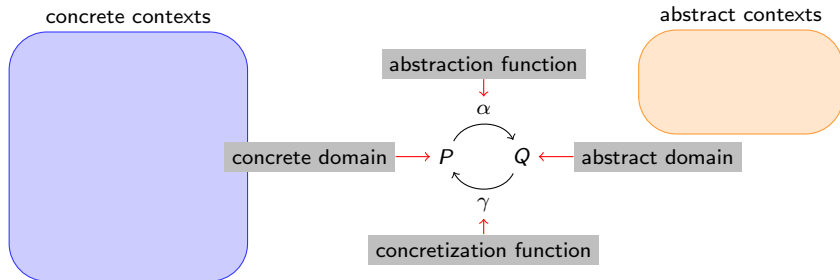
$$p \leq \gamma(q) \iff \alpha(p) \sqsubseteq q$$

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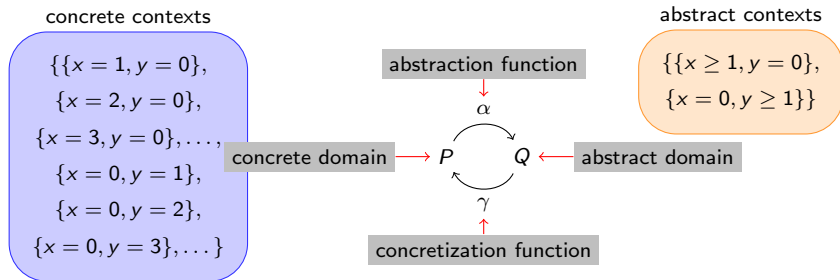


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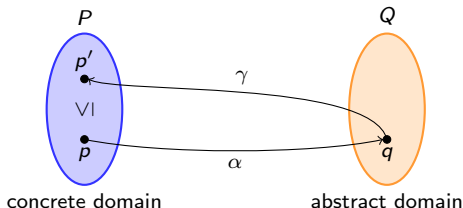


Galois Connections

- **Implication:** if abstraction and concretization functions of an abstract interpretation form a Galois connection, the abstract interpretation may only over-approximate (never under-approximate) \Rightarrow it is **sound**.

Proof.

$$\begin{aligned} \alpha(p) \sqsubseteq q &\iff p \leq \gamma(q) \implies \\ \Rightarrow \alpha(p) = q &\implies p \leq \gamma(q) \implies \\ \Rightarrow \alpha(p) = q &\implies p \leq \gamma(\alpha(p)) \end{aligned}$$



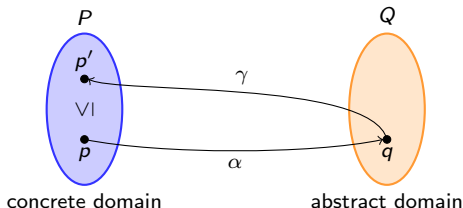
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- Moreover, each instruction i from Instr and a corresponding abstract transformer τ_i need to respect the Galois connection:

$$\alpha(i(p_1, \dots, p_n)) \sqsubseteq \tau_i(\alpha(p_1), \dots, \alpha(p_n)).$$

Fixpoint Approximation

- In some cases (e.g., loops), computation of the *most precise* abstract fixpoint is not generally guaranteed to terminate (consider *id* as the abstraction function).
- To guarantee termination, the fixpoint can be approximated. This is done using the following two operations:
 - **widening**: performs an over-approximation of a fixpoint,
 - **narrowing**: refines an approximation of a fixpoint.
- Neither widening nor narrowing are necessary, but at least widening is often convenient. Narrowing may be sometimes missing (e.g., in polyhedral analysis).

Widening

- Let $I = (Q, \circ, \sqsubseteq, \top, \perp, \tau)$ be an abstract interpretation of a program.
- The binary **widening** operation ∇ is defined as:
 - $\nabla : Q \times Q \rightarrow Q$,
 - $\forall C, D \in Q : (C \circ D) \sqsubseteq (C \nabla D)$,
 - for all infinite sequences $(C_0, \dots, C_n, \dots) \in Q^\omega$, it holds that the infinite sequence (s_0, \dots, s_n, \dots) defined recursively as

$$\begin{aligned}s_0 &= C_0, \\ s_n &= s_{n-1} \nabla C_n\end{aligned}$$

is not strictly increasing (and because the result of ∇ is an upper bound, the sequence eventually stabilizes).

- Widening can be applied later in the computation, the later it is applied the more precise is the result (but the computation takes longer time).

Narrowing

- Let $I = (Q, \circ, \sqsubseteq, \top, \perp, \tau)$ be an abstract interpretation of a program.
- The binary **narrowing** operation Δ is defined as:
 - $\Delta: Q \times Q \rightarrow Q$,
 - $\forall C, D \in Q : C \sqsupseteq D \Rightarrow (C \sqsupseteq (C \Delta D) \sqsupseteq D)$,
 - for all infinite sequences $(C_0, \dots, C_n, \dots) \in Q^\omega$, it holds that the infinite sequence (s_0, \dots, s_n, \dots) defined recursively as

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is not strictly decreasing (and because the result of $C \Delta D$ is a lower bound of C , the sequence eventually stabilizes provided that the input sequence is not strictly increasing).

- Narrowing is performed only after widening.

Representation of a Program

- We choose (deterministic) **finite flowcharts** as a language independent representation of programs.
- Finite flowchart is a directed graph with 5 types of nodes:
 - entries,
 - assignments,
 - tests,
 - junctions,
 - exits.
- Abstract interpretation iteratively computes abstract contexts for each edge of the flowchart.
- An equation is associated with each edge of the flowchart according to the type of the tail node of the edge.

Representation of a Program

- **Entry**: denotes the entry point of a program. $C_O = \top$

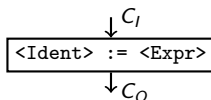


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 $C_O = \tau(A, C_I)$

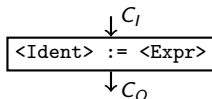


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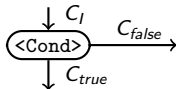
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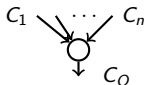


- **Test:** denotes splitting of the flow to branches B_{true} and B_{false} according to the Boolean condition $\langle \text{Cond} \rangle$. Two context are computed: $C_{true} = \tau(B_{true}, C_I)$ and $C_{false} = \tau(B_{false}, C_I)$



Representation of a Program

- **Junction**: denotes join J of several branches of code execution (e.g., after `...then ...` and `...else ...` branches of an if statement or for a loop join).
 $C_O = \tau(J, C_1 \circ \dots \circ C_n)$



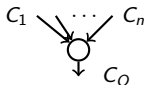
It often holds for junctions that

- $\tau(J) = \lambda x . x$ — for simple junctions (if branches),
- $\tau(J) = \lambda x . C_p \nabla x$ — for loop junctions,
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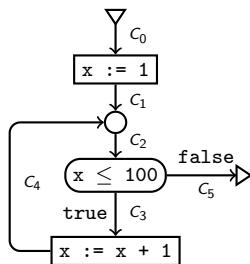
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- **Exit**: denotes the exit point of a program.



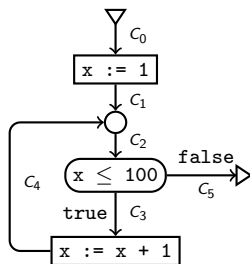
Program Example

- Consider the following flowchart program and [interval](#) analysis:



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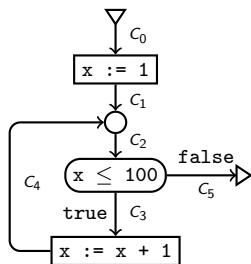
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- Assignments** are treated using an integer arithmetic (e.g., $[i, j] + [k, l] = [i + k, j + l]$).
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- We define the **widening** ∇ of intervals as:
 - $[,]$ is the null element of ∇ ,
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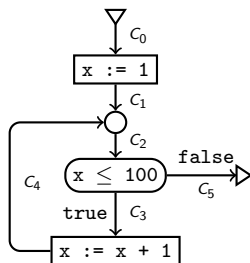
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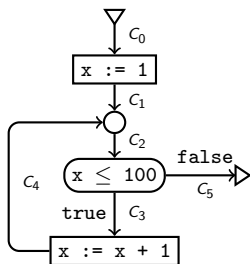
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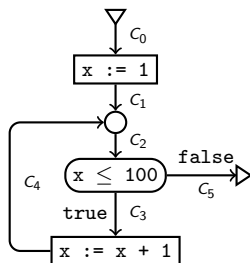
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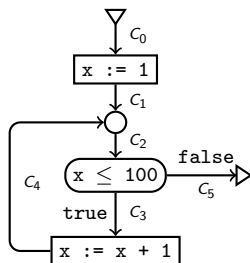
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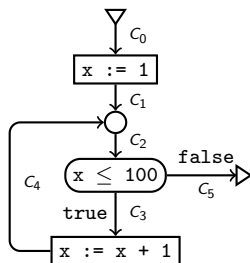
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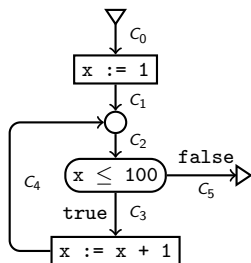
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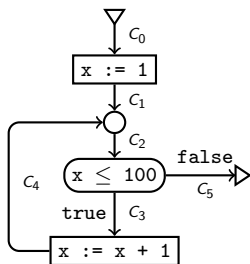
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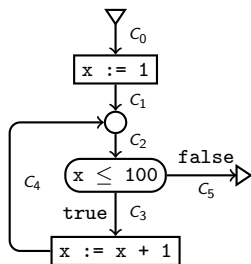
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 - $[i, j] \nabla [k, l] = [\text{if } k < i \text{ then } -\infty \text{ else } i, \text{if } l > j \text{ then } +\infty \text{ else } j]$.

- $C_0^0 = [-\infty, +\infty]$
- $C_1^0 = [,]$ $C_1^1 = [1, 1]$
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- $C_3^0 = [,]$ $C_3^1 = [1, 1]$ $C_3^2 = [1, 100]$
- $C_4^0 = [,]$ $C_4^1 = [2, 2]$
- $C_5^0 = [,]$

- $C_0 = [-\infty, +\infty]$
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- $C_2 = C_2 \nabla (C_1 \cup C_4)$
- $C_3 = C_2 \cap [-\infty, 100]$
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- $C_5 = C_2 \cap [101, +\infty]$

Program Example

- Consider the following flowchart program and **interval** analysis:



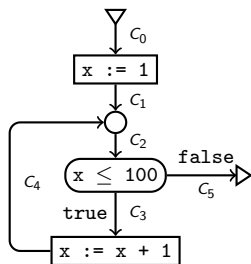
- We will use notation $[a, b]$ for the predicate $a \leq x \leq b$.
- Assignments** are treated using an integer arithmetic (e.g., $[i, j] + [k, l] = [i + k, j + l]$).
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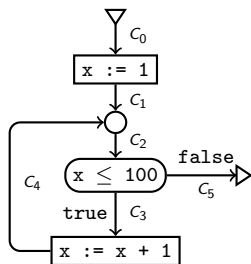
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Program Example

- Consider the following flowchart program and **interval** analysis:



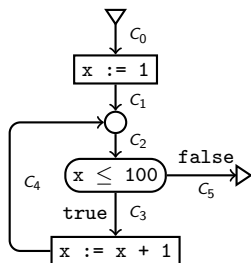
- Let us now define the **narrowing** operation Δ over intervals as
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 - $[i, j] \Delta [k, l] = [$
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- We now substitute the equation for C_2 with a new one that uses narrowing.

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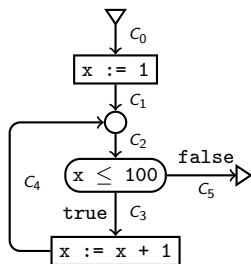
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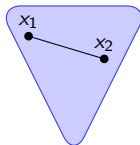
Examples of Abstract Interpretation

- **Interval analysis**: represents values of variables by intervals of possible values.
- **Polyhedral analysis**: represents values of variables by a convex polyhedron (a system of linear inequalities). This can be used to discover invariants of programs.
 - APRON (also intervals, octagons, etc.), ...
- **Heap analysis**: overapproximates graphs representing the memory heap. This can be used, e.g., for memory leak detection.
 - CINV, Forester, Predator, Space Invader, ...
- **Worst-case execution time (WCET) analysis**: may involve the analysis of the cache behaviour, the pipelines, etc.
 - AbsInt, ...
- and many more ...
 - Astrée, ECLAIR, Fluctuat, Polyspace, Stanford Checker, ...

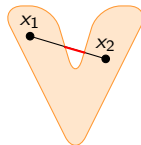
Polyhedral Analysis

Polyhedral Analysis

- An abstract interpretation-based approach of automatic discovering of relations among program variables expressible as **linear inequations**
 - this can be seen as a generalization of the interval analysis.
- Let $\vec{x} = x_1, \dots, x_n \in \mathbb{R}^n$ be the variables of a program. We can use a **convex polyhedron** $P \subseteq \mathbb{R}^n$ to represent a set of assignments to \vec{x} .
- We use **convex** polyhedra because operations on them are reasonably efficient (a set $C \subseteq \mathbb{R}^n$ is convex iff $\forall x_1, x_2 \in C, \forall 0 \leq \lambda \leq 1 : \lambda x_1 + (1 - \lambda)x_2 \in C$).



convex set



non-convex set

- **Use:** compile-time determination of bounds of variables, discovery of constants, ...

Representation of a Convex Polyhedron

- We use two dual ways to represent a convex polyhedron:
 - by a **system of linear inequations**, and
 - by the **frame of the polyhedron**.
- We can alter between these representations (with some overhead).
- Efficient execution of different operations require different representation.

System of Linear Inequations

- Let $\vec{x} = x_1, \dots, x_n \in \mathbb{R}^n$ be the variables of a program. Given a finite set of m linear inequations over \vec{x} of the form

$$\left\{ \sum_{i=1}^n a_{ji} x_i \leq b_j \mid 1 \leq j \leq m \right\}$$

or equivalently using vectors and matrices as

$$\vec{x} \cdot \mathbf{A} \leq \vec{b}$$

we can geometrically interpret the solutions of the inequations as a **convex polyhedron** in \mathbb{R}^n defined by the intersection of *halfspaces* corresponding to each inequality.

The Frame of a Convex Polyhedron

- A convex polyhedron P can also be characterized by its **frame** $F = (V, R, L)$:
 - **vertices** V : points \vec{v} of a polyhedron P which are not *convex combinations* of other points $\{\vec{w}_1, \dots, \vec{w}_m\}$ of P ,

$$\left(\left(\vec{v} = \sum_{i=1}^m \lambda_i \vec{w}_i \right) \wedge (\forall 1 \leq i \leq m : (\vec{w}_i \in P \wedge \lambda_i \geq 0)) \wedge \left(\sum_{i=1}^m \lambda_i = 1 \right) \right) \Rightarrow \\ \Rightarrow (\forall 1 \leq i \leq m : (\lambda_i = 0 \vee \vec{w}_i = \vec{v}))$$

- **Convex hull**: the set of all convex combinations of V .

The Frame of a Convex Polyhedron

- **extreme rays** R : rays \vec{r} of P (i.e. vectors such that there exists a half-line parallel to \vec{r} and entirely included in P) which are not positive combinations of other rays $\vec{s}_1, \dots, \vec{s}_p$ of P :

$$\left(\vec{r} = \sum_{i=1}^p \mu_i \vec{s}_i \wedge (\forall 1 \leq i \leq p : \mu_i \in \mathbb{R}^+) \right) \Rightarrow (\forall 1 \leq i \leq p : (\mu_i = 0 \vee \vec{s}_i = \vec{r}))$$

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- **lines** L : vectors \vec{l} such that both \vec{l} and $-\vec{l}$ are rays of P :

$$\forall \vec{x} \in P, \forall \mu \in \mathbb{R} : \vec{x} + \mu \vec{l} \in P$$

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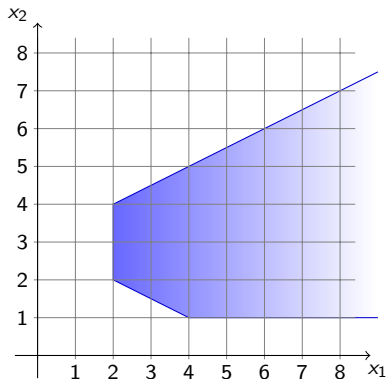
$$\forall \vec{x} \in P, \forall \mu \in \mathbb{R} : \vec{x} + \mu \vec{l} \in P$$

- Every point \vec{x} of the polyhedron P defined by the frame $F = (V, R, L)$ can be obtained from V , R and L :

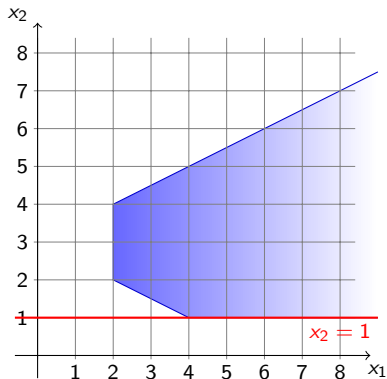
$$\vec{x} = \sum_{i=1}^{\sigma} \lambda_i \vec{v}_i + \sum_{j=1}^{\rho} \mu_j \vec{r}_j + \sum_{k=1}^{\delta} \nu_k \vec{l}_k$$

$$\text{where } 0 \leq \lambda_1, \dots, \lambda_{\sigma} \leq 1, \sum_{i=1}^{\sigma} \lambda_i = 1, \mu_1, \dots, \mu_{\rho} \in \mathbb{R}^+, \nu_1, \dots, \nu_{\delta} \in \mathbb{R}$$

Example of a Convex Polyhedron



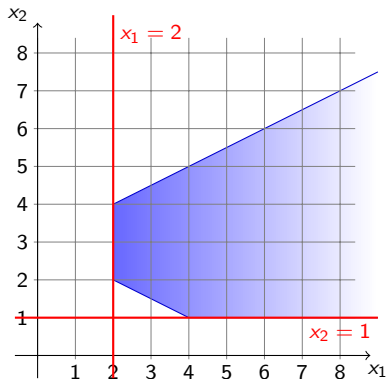
Example of a Convex Polyhedron



System of linear inequations

$$x_2 \geq 1$$

Example of a Convex Polyhedron

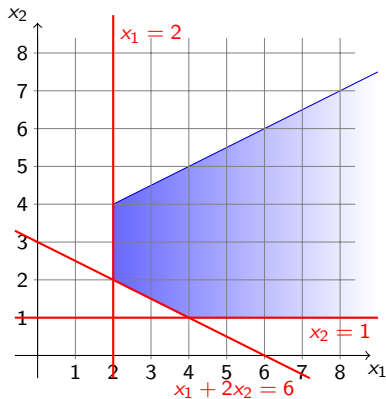


System of linear inequations

$$x_2 \geq 1$$

$$x_1 \geq 2$$

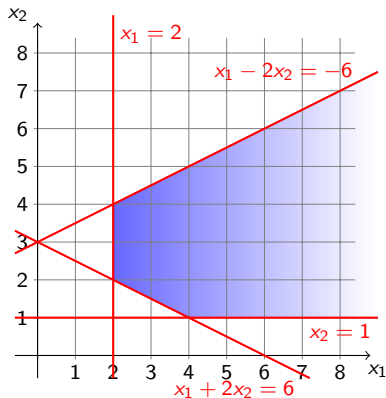
Example of a Convex Polyhedron



System of linear inequations

$$\begin{aligned}x_2 &\geq 1 \\x_1 &\geq 2 \\x_1 + 2x_2 &\geq 6\end{aligned}$$

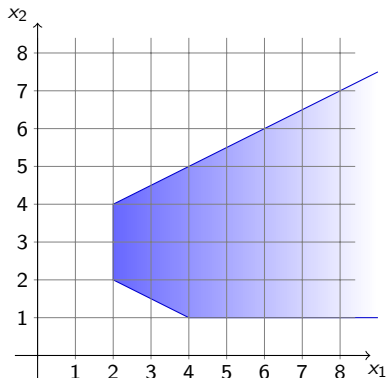
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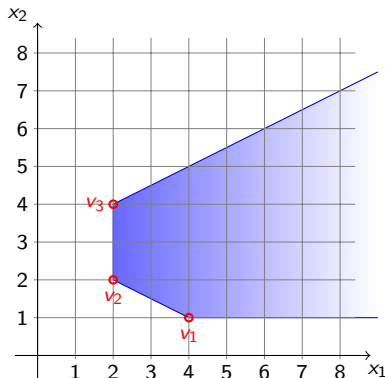
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Frame of a polyhedron

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Example of a Convex Polyhedron



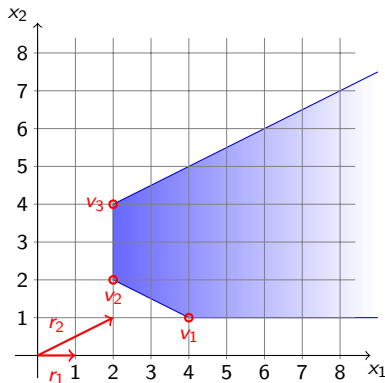
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Frame of a polyhedron

$$\begin{aligned}F &= (V, R, L) \\V &= \{\vec{v}_1 = [4, 1], \vec{v}_2 = [2, 2], \vec{v}_3 = [2, 4]\}\end{aligned}$$

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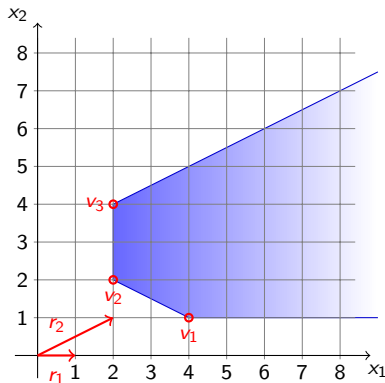
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Transformations of Convex Polyhedra

- Different types of nodes of the flowchart representation of a program perform distinct transformation on the polyhedron. The number of input and output polyhedra differs according to the type of the node.
- **Entries:** create a polyhedron according to restraints on the input values of variables (in case there are none for variable x_i , the polyhedron is unbounded in i -th dimension).

Assignments

- Performed operations vary according to assigned expression:
 - **non-linear expression** $x_i := \langle \text{non-linear expression} \rangle$: because these cannot be represented using convex polyhedra, any restraint on x_i is dropped (we add line \vec{d} to frame such that $d_i = 1$ and $\forall 1 \leq j \leq n, i \neq j : d_j = 0$).
 - **linear expression** $x_i := \sum_{j=1}^n a_j x_j + b$: the frame $F' = (V', R', L')$ of the output polyhedron can be computed from the frame $F = (V, R, L)$ of the input as

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 - $L' = \{\vec{l}'_1, \dots, \vec{l}'_\delta\}$ where \vec{l}'_j is defined by $l'_{ji} = \vec{a} \vec{l}_j$ and $l'_{jm} = l_{jm}$ for $\forall 1 \leq m \leq \delta, m \neq j$.

Tests

- The input polyhedron P is transformed into two output polyhedra: P_t for the true branch and P_f for the false branch.
- For a Boolean condition C it needs to hold that $P_t \supseteq P \cap T_C$, $P_f \supseteq P \setminus T_C$ where T_C is the subset of \mathbb{R}^n such that each point of T_C satisfies C (these are not necessarily convex polyhedra).
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- The operation that is performed varies according to the Boolean condition of the test:
 - **Non-linear tests:** $P_t = P_f = P$ (can be refined for some cases)
 - **Linear equality tests:** Boolean condition $C : \vec{a}\vec{x} = b$ defines a *hyperplane* H . If P is included in H then $P_t = P$, $P_f = \emptyset$. If P is not included in H then $P_t = P \cap H$ and $P_f = P$.

Tests

- The input polyhedron P is transformed into two output polyhedra: P_t for the true branch and P_f for the false branch.
- For a Boolean condition C it needs to hold that $P_t \supseteq P \cap T_C$, $P_f \supseteq P \setminus T_C$ where T_C is the subset of \mathbb{R}^n such that each point of T_C satisfies C (these are not necessarily convex polyhedra).
- The operation that is performed varies according to the Boolean condition of the test:
 - **Non-linear tests:** $P_t = P_f = P$ (can be refined for some cases)
 - **Linear equality tests:** Boolean condition $C : \vec{a}\vec{x} = b$ defines a *hyperplane* H . If P is included in H then $P_t = P$, $P_f = \emptyset$. If P is not included in H then $P_t = P \cap H$ and $P_f = P$.
 - **Linear inequality tests:** for Boolean condition $\vec{a}\vec{x} \leq b$ the outputs are $P_t = P \cap \vec{a}\vec{x} \leq b$ and $P_f = P \cap \vec{a}\vec{x} \geq b$.

Junctions

- Junctions correspond to merge of several program paths so the output polyhedron P is union of all input polyhedra P_i . It is computed according to the kind of the junction:

Junctions

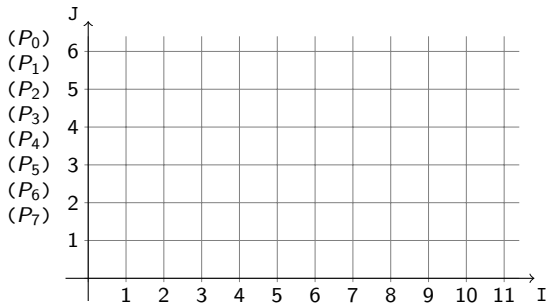
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 - **Simple junctions:** for input polyhedra P_1, \dots, P_m we compute the convex hull of $P_1 \cup \dots \cup P_m$

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 - **Simple junctions:** for input polyhedra P_1, \dots, P_m we compute the convex hull of $P_1 \cup \dots \cup P_m$
 - **Loop junctions:** for input polyhedra P_1, \dots, P_m let Q be the convex hull of $P_1 \cup \dots \cup P_m$. Then $P' = P \nabla Q$ is the convex polyhedron consisting of linear restraints of P satisfied by every element of Q .

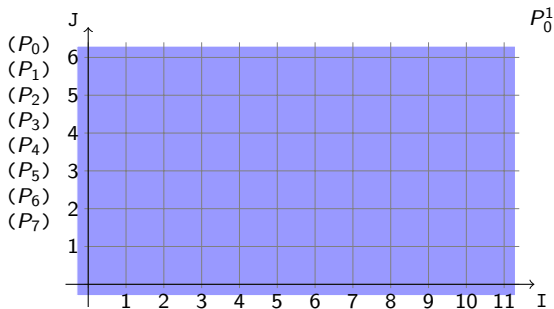
Example

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I := 2, J := 0;  
L:  
  if ... then  
    I := I + 4  
  else  
    J := J + 1, I := I + 2;  
  fi;  
go to L;
```



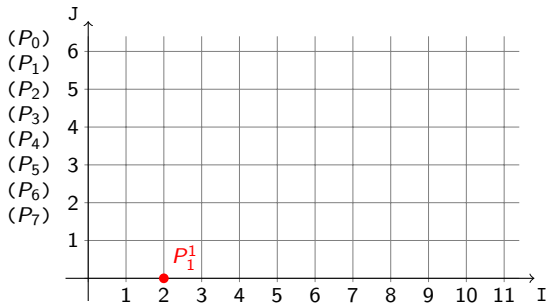
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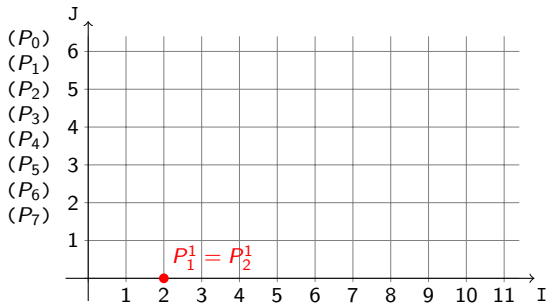
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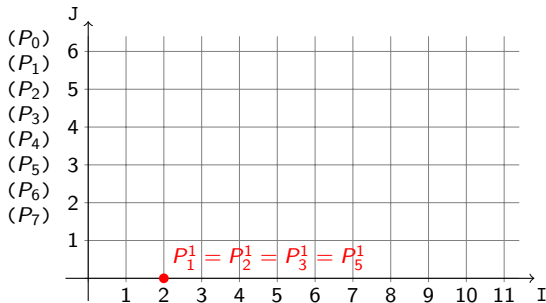
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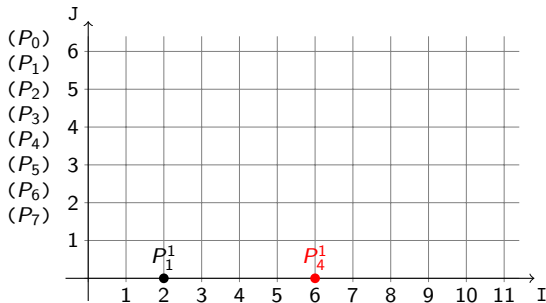
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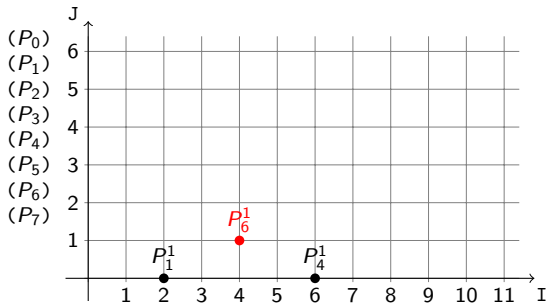
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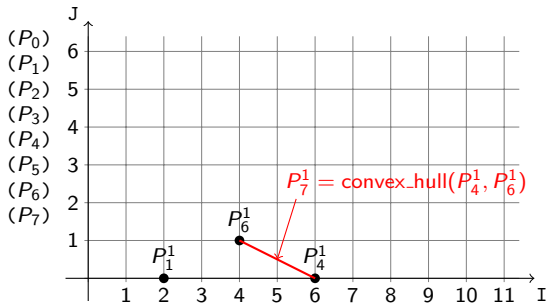
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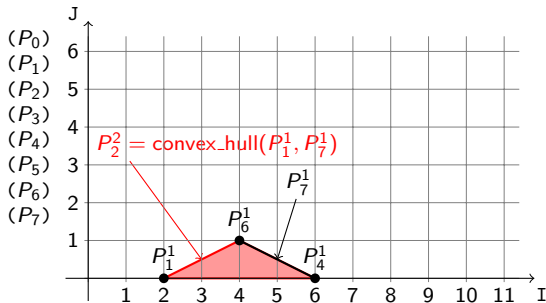
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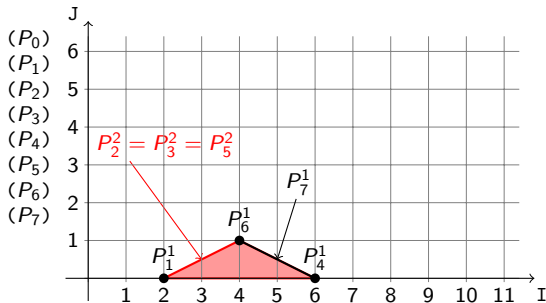
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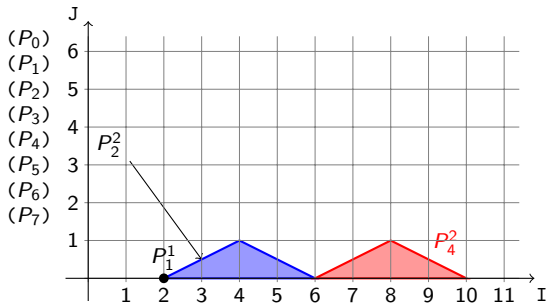
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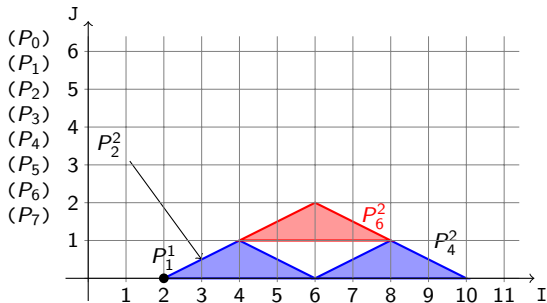
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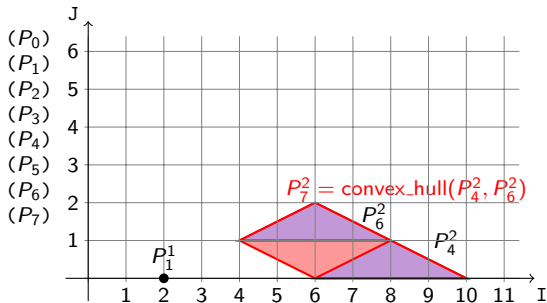
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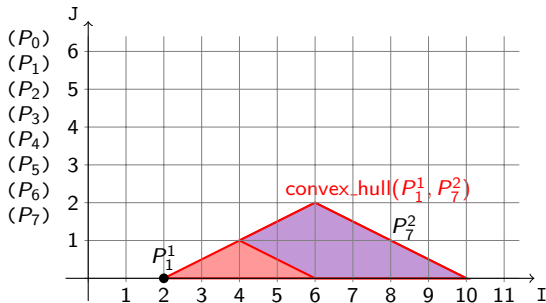
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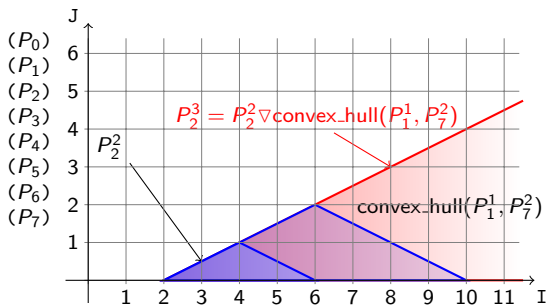
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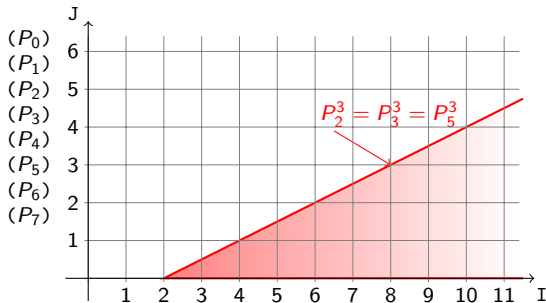
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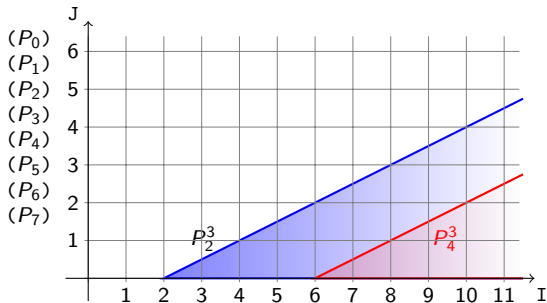
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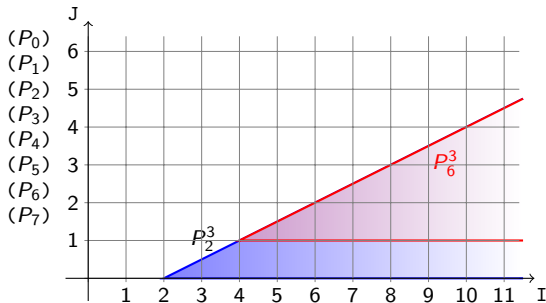
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