Formal Analysis and Verification

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Abstract Interpretation

Introduction

- Compared to model checking in which the stress is put on a systematic execution of a system being verified (or its model), the emphasis in static analysis is on minimization of the amount of execution of the code. It is either not executed at all (the case of looking for bug patterns) or just on some abstract level, typically with an in advance fixed abstraction (data flow analysis, abstract interpretation, ...).
- However, the borderline between model checking and static analysis is not sharp (especially when considering abstract interpretation and model checking based on predicate abstraction).
- Many static analyses are such that they can be applied to parts of code without the need to describe their environment.
- Static analysis approaches: bug pattern analysis, type analysis, data flow analysis, ..., abstract interpretation, (and sometimes even model checking).

Static Analyses

- Many different classes of programs:
  - control-intensive programs,
  - digital signal processing,
  - programs manipulating dynamic memory,
  - programs with integers, . . .
- To make an analysis efficient (effective), some form of abstraction is often needed.
- An analysis successful for one class may (and is very likely to) fail for a different one (too much imprecision, inefficiency, divergence).
- Analyses are tailored for specific classes of programs
  - the need to prove soundness (completeness) of each analysis.

Abstract Interpretation

- Introduced by Patrick and Radhia Cousot at POPL'77.
- A general framework for static analyses.
- Concrete analyses are created by providing specific components (abstract domain, abstract transformers, . . .) to the framework.
- Abstract interpretation transforms a program into an abstract program over an abstract domain and analyses this (cf. predicate abstraction).
- When certain properties of the components are met, the analysis is guaranteed to be sound.

Ingredients of Abstract Interpretation

- Abstract domain
  - program states at program locations are represented using abstract contexts.
- Abstract transformers
  - for each program operation there is a corresponding transformer that represents the effect of the operation performed on an abstract context.
- Join operator
  - combines abstract contexts from several branches into a single one.
- Widening
  - performed on a sequence of abstract contexts appearing at a given location to accelerate obtaining a fixpoint.
- Narrowing
  - may be used to refine the result of widening.

Abstract Interpretation — formally

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- A semilattice is a poset with a join (least upper bound, supremum) for every finite subset of its base set.
- Abstract interpretation I of a program P with the instruction set `Instr` is a tuple

$$I = (Q, \circ, \sqsubseteq, \top, \bot, \tau)$$

where

  - Q is the abstract domain (domain of abstract contexts),
  - $\top \in Q$ is the supremum of Q,
  - $\bot \in Q$ is the infimum of Q,
  - $\circ : Q \times Q \rightarrow Q$ is the join operator for accumulation of abstract contexts, $(Q, \circ, \top)$ is a complete semilattice,
  - $(\sqsubseteq) \subseteq Q \times Q$ is an ordering defined as $x \sqsubseteq y \iff x \circ y = y$ in $(Q, \circ, \top)$,
  - $\tau : \texttt{Instr} \times Q \rightarrow Q$ defines the interpretation of abstract transformers.

- The soundness of abstract interpretation is guaranteed using Galois connections.

Galois Connections

- Galois connection is a quadruple $\pi = (\mathcal{P}, \alpha, \gamma, \mathcal{Q})$ such that:
  - $\mathcal{P} = \langle P, \leq \rangle$ and $\mathcal{Q} = \langle Q, \sqsubseteq \rangle$ are *partially ordered sets* (posets),
  - $\alpha : P \to Q$ and $\gamma : Q \to P$ are functions such that $\forall p \in P$ and $\forall q \in Q$:

$$p \leq \gamma(q) \iff \alpha(p) \sqsubseteq q$$

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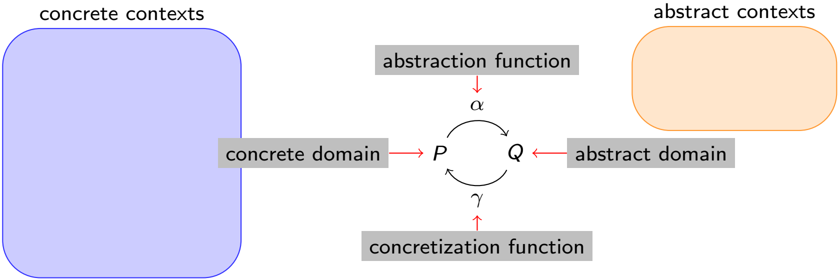

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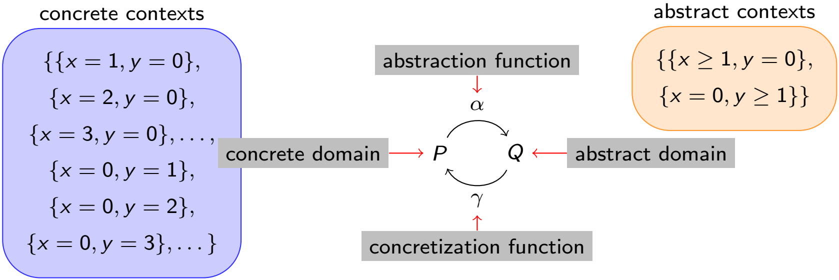

Galois Connections

- **Implication**: if abstraction and concretization functions of an abstract interpretation form a Galois connection, the abstract interpretation may only over-approximate (never under-approximate) $\Rightarrow$ it is sound.

Proof.

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\begin{array}{cccccc}
 & \alpha(p) \sqsubseteq q & \Longleftrightarrow & p \leq \gamma(q) & \Rightarrow \\
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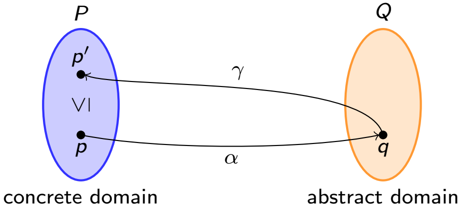

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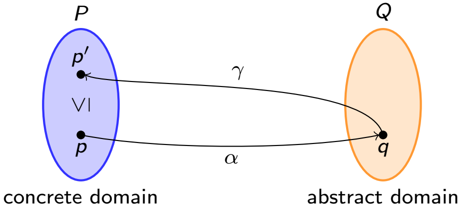

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- Moreover, each instruction i from `Instr` and a corresponding abstract transformer τ_i need to respect the Galois connection:

$$\alpha(i(p_1, \ldots, p_n)) \sqsubseteq \tau_i(\alpha(p_1), \ldots, \alpha(p_n)).$$

Fixpoint Approximation

- In some cases (e.g., loops), computation of the *most precise* abstract fixpoint is not generally guaranteed to terminate (consider *id* as the abstraction function).
- To guarantee termination, the fixpoint can be approximated. This is done using the following two operations:
  - widening: performs over-approximation of a fixpoint,
  - narrowing: refines approximation of a fixpoint.
- Neither widening nor narrowing are necessary, but at least widening is often convenient. Narrowing may be sometimes missing (e.g., in polyhedral analysis).

Widening

- Let $I = (Q, \circ, \sqsubseteq, \top, \bot, \tau)$ be an abstract interpretation of a program.
- The binary widening operation $\triangledown$ is defined as:
  - $\triangledown : Q \times Q \to Q$,
  - $\forall C, D \in Q : (C \circ D) \sqsubseteq (C \triangledown D)$,
  - for all infinite sequences $(C_0, \ldots, C_n, \ldots) \in Q^\omega$, it holds that the infinite sequence $(s_0, \ldots, s_n, \ldots)$ defined recursively as

    $$\begin{aligned} s_0 &= C_0, \\ s_n &= s_{n-1} \triangledown C_n \end{aligned}$$

    is not strictly increasing (and because the result of $\triangledown$ is an upper bound, the sequence eventually stabilizes).
- Widening can be applied later in the computation, the later it is applied the more precise is the result (but the computation takes longer time).

Narrowing

- Let $I = (Q, \circ, \sqsubseteq, \top, \bot, \tau)$ be an abstract interpretation of a program.
- The binary narrowing operation $\triangle$ is defined as:
  - $\triangle$: $Q \times Q \to Q$,
  - $\forall C, D \in Q : C \sqsupseteq D \Rightarrow (C \sqsupseteq (C \triangle D) \sqsupseteq D)$,
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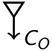
    is not strictly decreasing (and because the result of $C \triangle D$ is a lower bound of C, the sequence eventually stabilizes provided that the input sequence is not strictly increasing).
- Narrowing is performed only after widening.

Representation of a Program

- We choose (deterministic) finite flowcharts as a language independent representation of programs.
- Finite flowchart is a directed graph with 5 types of nodes:
  - entries,
  - assignments,
  - tests,
  - junctions,
  - exits.
- Abstract interpretation iteratively computes abstract contexts for each edge of the flowchart.
- An equation is associated with each edge of the flowchart according to the type of the tail node of the edge.

Representation of a Program

- Entry: denotes the entry point of a program. $C_O = \top$



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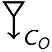
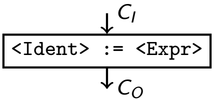


- Assignment: denotes the assignment A of expression `<Expr>` to the variable `<Ident>`.
  $C_O = \tau(A, C_I)$



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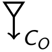

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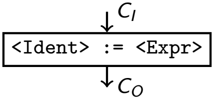
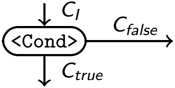
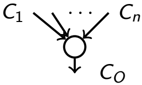


- Test: denotes splitting of the flow to branches B_{true} and B_{false} according to the Boolean condition `<Cond>`. Two context are computed: $C_{true} = \tau(B_{true}, C_I)$ and $C_{false} = \tau(B_{false}, C_I)$



Representation of a Program

- Junction: denotes join J of several branches of code execution (e.g., after `...then ...` and `...else ...` branches of an `if` statement or for a loop join).
  $C_O = \tau(J, C_1 \circ \cdots \circ C_n)$



  It often holds for junctions that
  - $\tau(J) = \lambda x \,.\, x$ — for simple junctions (`if` branches),
  - $\tau(J) = \lambda x \,.\, C_p \triangledown x$ — for loop junctions,
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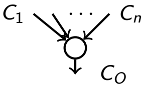

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- Exit: denotes the exit point of a program.

Program Example

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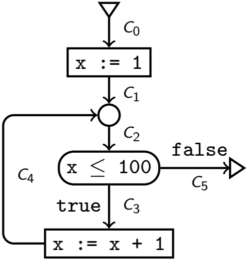

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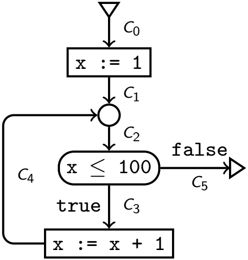

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- Assignments are treated using an integer arithmetic (e.g., $[i, j] + [k, l] = [i + k, j + l]$).
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- We define the widening $\triangledown$ of intervals as:
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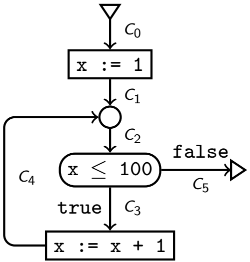

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- $C_1 = [1, 1]$
- $C_2 = C_2 \nabla (C_1 \cup C_4)$
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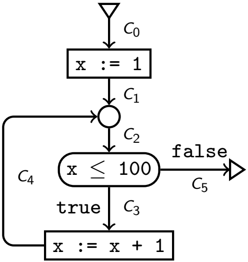

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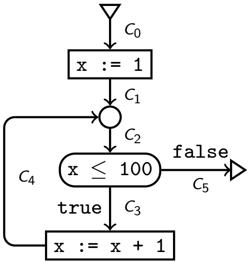

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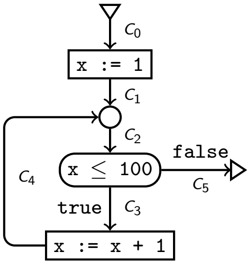

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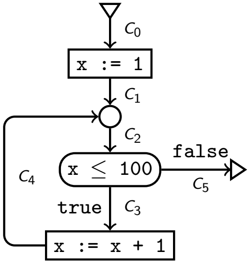

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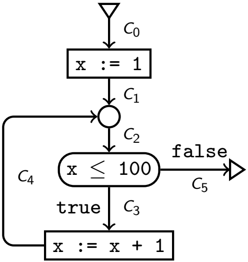

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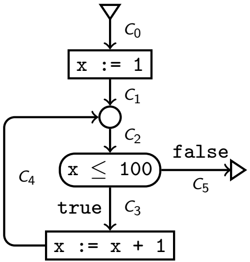

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Program Example

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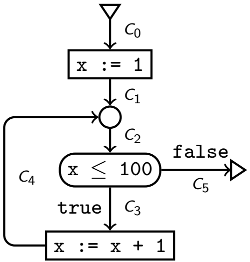

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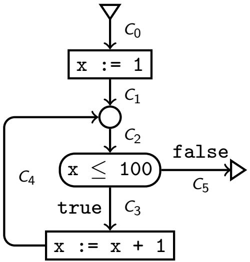

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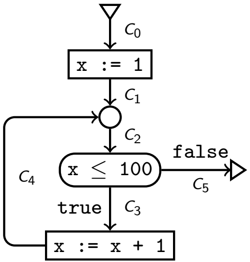

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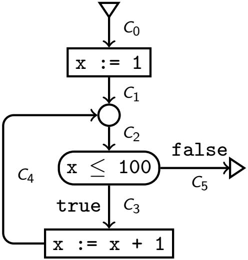

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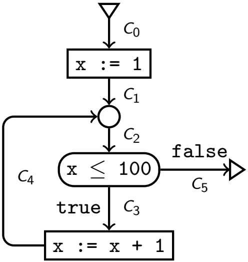

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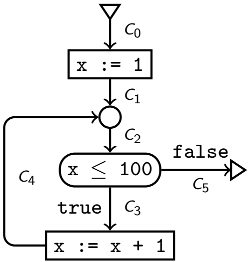

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Examples of Abstract Interpretation

- Interval analysis: represents values of variables by intervals of possible values.
- Polyhedral analysis: represents values of variables by a convex polyhedron (a system of linear inequalities). This can be used to discover invariants of programs.
    - APRON (also intervals, octagons, etc.), ...
- Heap analysis: overapproximates graphs representing the memory heap. This can be used, e.g., for memory leak detection.
    - CINV, Forester, Predator, Space Invader, ...
- Worst-case execution time (WCET) analysis: may involve the analysis of the cache behaviour, the pipelines, etc.
    - AbsInt, ...
- and many more ...
    - Astrée, ECLAIR, Fluctuat, Polyspace, Stanford Checker, ...

Polyhedral Analysis

Polyhedral Analysis

- An abstract interpretation-based approach of automatic discovering of relations among program variables expressible as linear inequations
  - this can be seen as a generalization of the interval analysis.
- Let $\vec{x} = x_1, \ldots, x_n \in \mathbb{R}^n$ be the variables of a program. We can use a convex polyhedron $P \subseteq \mathbb{R}^n$ to represent a set of assignments to $\vec{x}$.
- We use convex polyhedra because operations on them are reasonably efficient (a set $C \subseteq \mathbb{R}^n$ is convex iff $\forall x_1, x_2 \in C, \forall 0 \leq \lambda \leq 1 \ : \ \lambda x_1 + (1-\lambda)x_2 \in C$).

convex set　　non-convex set

- **Use:** compile-time determination of bounds of variables, discovery of constants, ...

Representation of a Convex Polyhedron

- We use two dual ways to represent a convex polyhedron:
  - by a system of linear inequations, and
  - by the frame of the polyhedron.
- We can alter between these representations (with some overhead).
- Efficient execution of different operations require different representation.

System of Linear Inequations

- Let $\vec{x} = x_1, \ldots, x_n \in \mathbb{R}^n$ be the variables of a program. Given a finite set of m linear inequations over $\vec{x}$ of the form

$$\left\{ \sum_{i=1}^{n} a_{ji} x_i \leq b_j \;\middle|\; 1 \leq j \leq m \right\}$$

or equivalently using vectors and matrices as

$$\vec{x} \cdot \mathbf{A} \leq \vec{b}$$

we can geometrically interpret the solutions of the inequations as a convex polyhedron in $\mathbb{R}^n$ defined by the intersection of *halfspaces* corresponding to each inequality.

The Frame of a Convex Polyhedron

- A convex polyhedron P can also be characterized by its frame $F = (V, R, L)$:
  - vertices V: points $\vec{v}$ of a polyhedron P which are not *convex combinations* of other points $\{\vec{w_1}, \ldots, \vec{w_m}\}$ of P,

$$\left(\left(\vec{v} = \sum_{i=1}^{m} \lambda_i \vec{w_i}\right) \wedge (\forall 1 \leq i \leq m : (\vec{w_i} \in P \wedge \lambda_i \geq 0)) \wedge \left(\sum_{i=1}^{m} \lambda_i = 1\right)\right) \Rightarrow$$

$$\Rightarrow (\forall 1 \leq i \leq m : (\lambda_i = 0 \vee \vec{w_i} = \vec{v}))$$

    - Convex hull: the set of all convex combinations of V.

The Frame of a Convex Polyhedron

- extreme rays R: rays $\vec{r}$ of P (i.e. vectors such that there exists a half-line parallel to $\vec{r}$ and entirely included in P) which are not positive combinations of other rays $\vec{s_1}, \ldots, \vec{s_p}$ of P:

$$\left(\vec{r} = \sum_{i=1}^{p} \mu_i \vec{s_i} \wedge \left(\forall 1 \leq i \leq p : \mu_i \in \mathbb{R}^+\right)\right) \Rightarrow (\forall 1 \leq i \leq p : (\mu_i = 0 \vee \vec{s_i} = \vec{r}))$$

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- lines L: vectors $\vec{l}$ such that both $\vec{l}$ and $-\vec{l}$ are rays of P:

$$\forall \vec{x} \in P, \forall \mu \in \mathbb{R} : \vec{x} + \mu \vec{l} \in P$$

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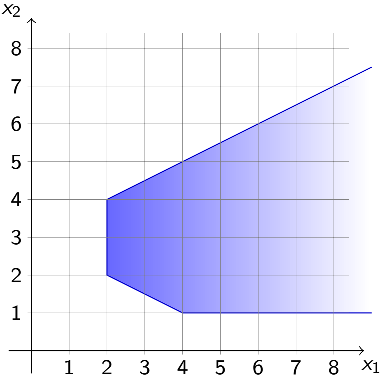
- Every point $\vec{x}$ of the polyhedron P defined by the frame $F = (V, R, L)$ can be obtained from V, R and L:

$$\vec{x} = \sum_{i=1}^{\sigma} \lambda_i \vec{v_i} + \sum_{j=1}^{\rho} \mu_j \vec{r_j} + \sum_{k=1}^{\delta} \nu_k \vec{l_k}$$

$$\text{where} \quad 0 \leq \lambda_1, \ldots, \lambda_\sigma \leq 1, \sum_{i=1}^{\sigma} \lambda_i = 1, \mu_1, \ldots, \mu_\rho \in \mathbb{R}^+, \nu_1, \ldots, \nu_\delta \in \mathbb{R}$$

Example of a Convex Polyhedron



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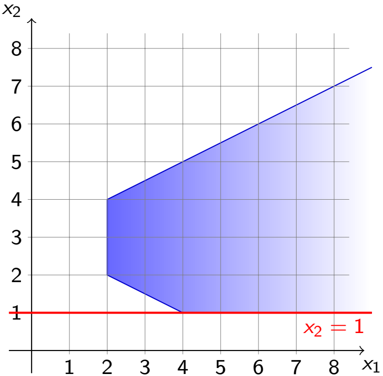

System of linear inequations

$$x_2 \geq 1$$

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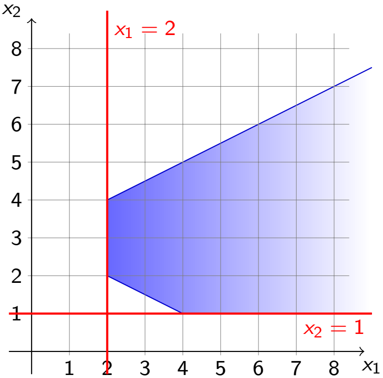

System of linear inequations

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\begin{aligned}
x_2 &\geq 1 \\
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\end{aligned}
$$

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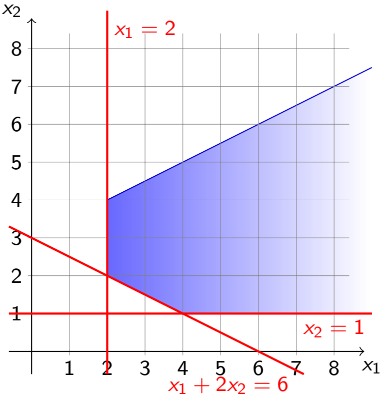

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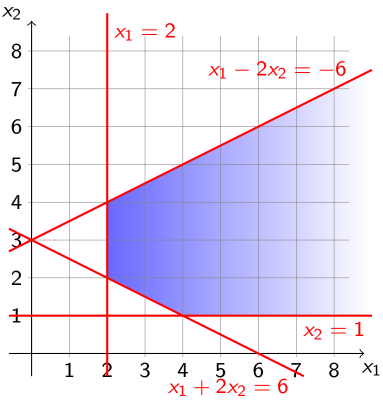

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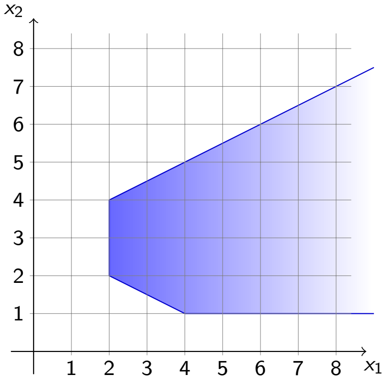

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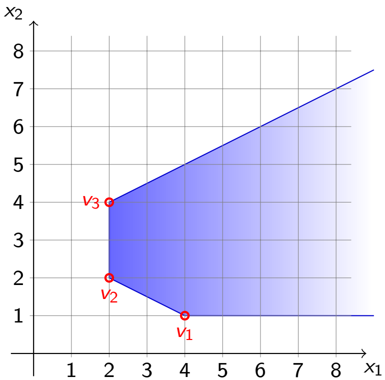

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F &= (V, R, L) \\
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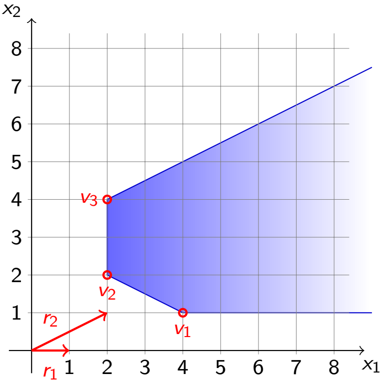

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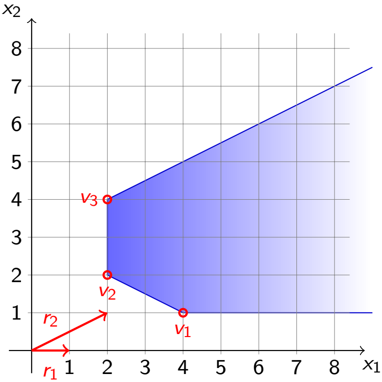

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L &= \emptyset
\end{aligned}
$$

Transformations of Convex Polyhedra

- Different types of nodes of the flowchart representation of a program perform distinct transformation on the polyhedron. The number of input and output polyhedra differs according to the type of the node.
- Entries: create a polyhedron according to restraints on the input values of variables (in case there are none for variable x_i, the polyhedron is unbounded in i-th dimension).

Assignments

- Performed operations vary according to assigned expression:
  - non-linear expression $x_i :=$ `<non-linear expression>`: because these cannot be represented using convex polyhedra, any restraint on x_i is dropped (we add line $\vec{d}$ to frame such that $d_i = 1$ and $\forall 1 \leq j \leq n, i \neq j : d_j = 0$).
  - linear expression $x_i := \sum_{j=1}^{n} a_j x_j + b$: the frame $F' = (V', R', L')$ of the output polyhedron can be computed from the frame $F = (V, R, L)$ of the input as

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    - $L' = \{\vec{l_1'}, \ldots, \vec{l_\delta'}\}$ where $\vec{l_j'}$ is defined by $l'_{ji} = \vec{a}\vec{l_j}$ and $l'_{jm} = l_{jm}$ for $\forall 1 \leq m \leq \delta, m \neq j$.

Tests

- The input polyhedron P is transformed into two output polyhedra: P_t for the `true` branch and P_f for the `false` branch.
- For a Boolean condition C it needs to hold that $P_t \supseteq P \cap T_C, P_f \supseteq P \setminus T_C$ where T_C is the subset of $\mathbb{R}^n$ such that each point of T_C satisfies C (these are not necessarily convex polyhedra).
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  - Loop junctions: for input polyhedra $P_1, \ldots, P_m$ let Q be the convex hull of $P_1 \cup \cdots \cup P_m$. Then $P' = P \triangledown Q$ is the convex polyhedron consisting of linear restraints of P satisfied by every element of Q.

Example

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I := 2, J := 0;
L:
if ... then
  I := I + 4
else
  J := J + 1, I := I + 2;
fi;
go to L;
```

(P_0)
(P_1)
(P_2)
(P_3)
(P_4)
(P_5)
(P_6)
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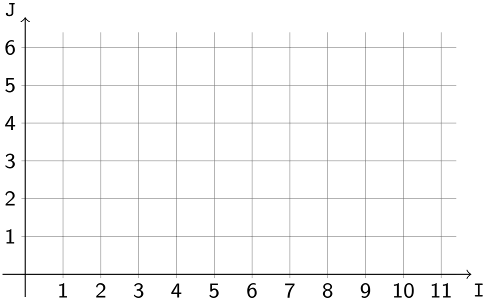

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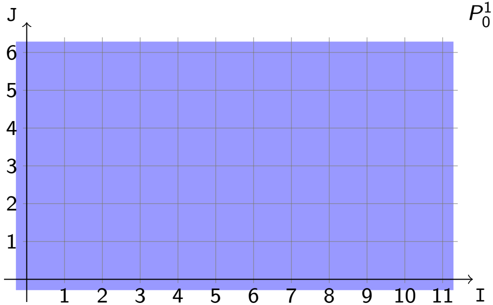

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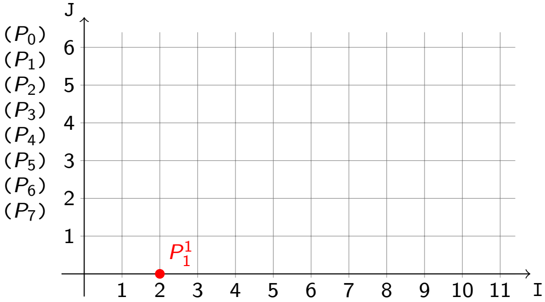

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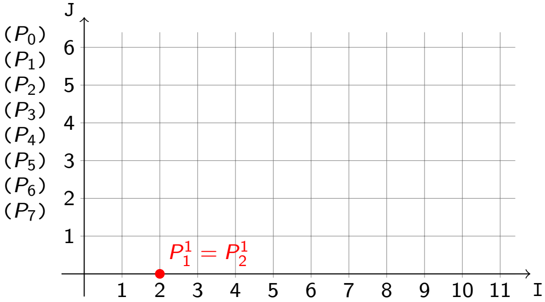

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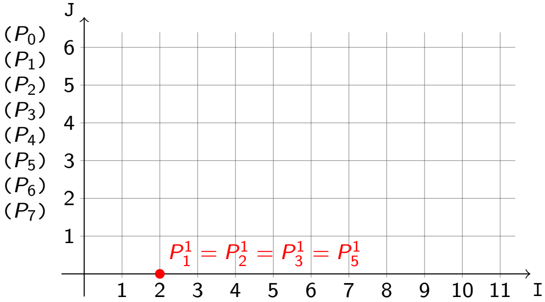

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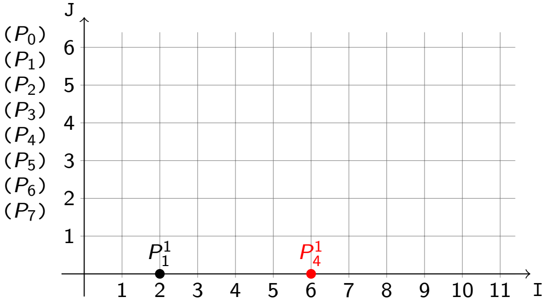

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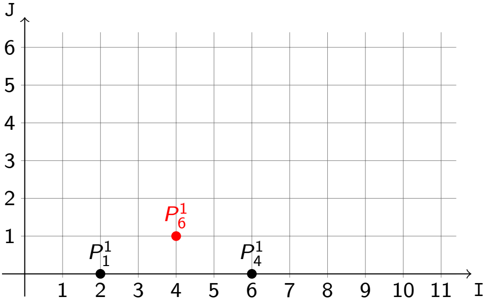

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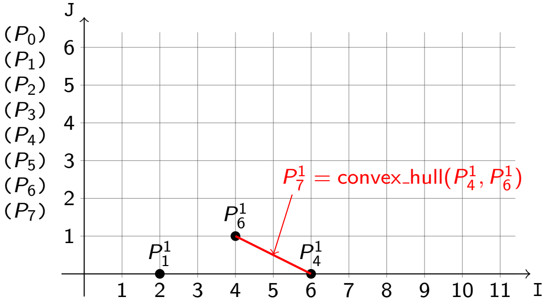

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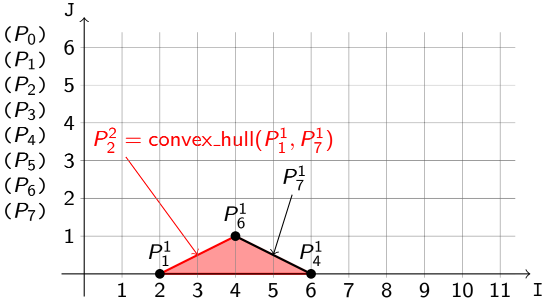

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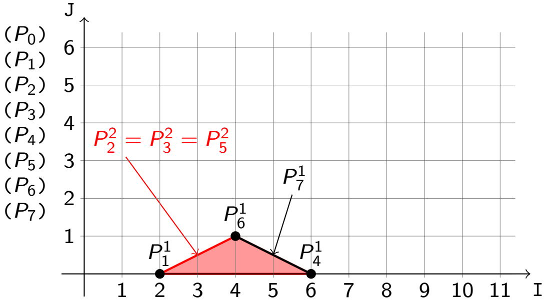

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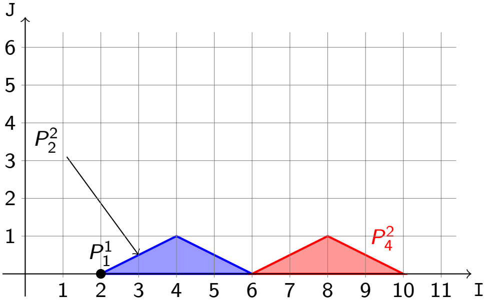

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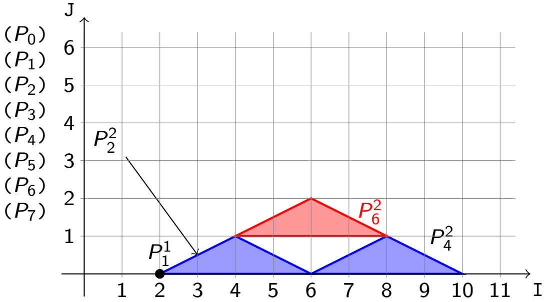

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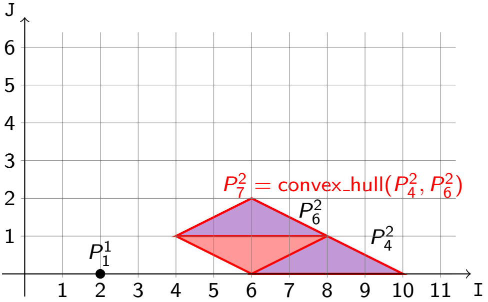

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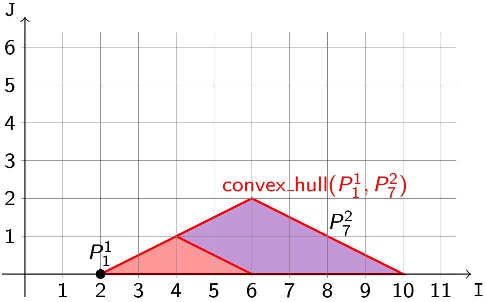

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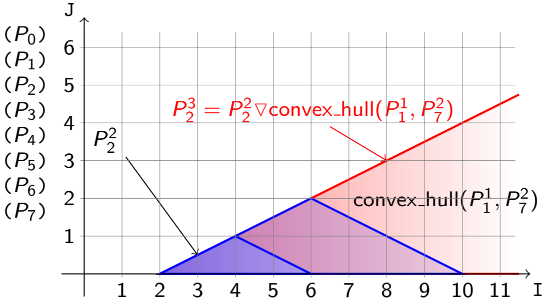

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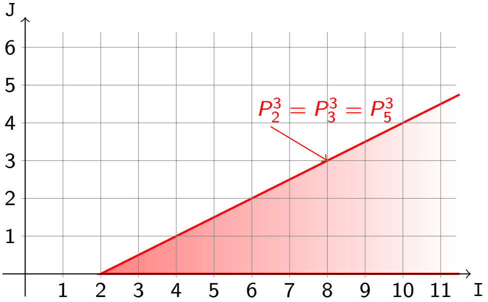

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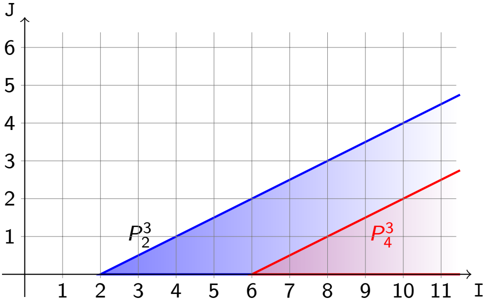

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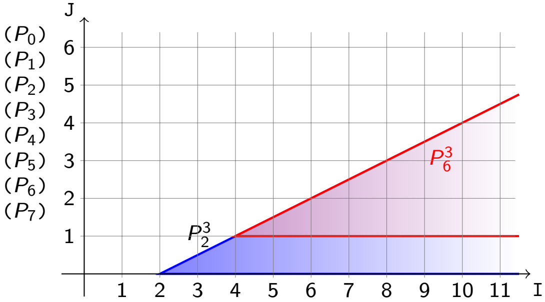

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